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FOREIGN TECHNOLOGY DIVISION

OPERATIONS RESEARCH

By

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5. QUEUEING THEORY.

1. Problems of queueing theory.

During operations research, very frequently we encounter with the analysis of the operaticn of the peculiar systems, called the systems of mass maintenance (SMO). As examples cf such systems can serve: exchanges, repair shop, booking cffices, reference bureaus, the stores, barbershop and the like.

Each SMO it consists of some number of operating unity which we will call the channels of maintenance. As "channels" can figure: the communication lines, operating points, instruments, railway lines, elevators, trucks, etc.

The systems of mass maintenance can be single-channel or multichannels.

Each SMO it is intended for maintenance (execution of some flow of claims (or "requirements"), that enter SMO at some, generally speaking, random moment of time. The maintenance of the acted claim

it is continued some (generally speaking, random) time, after which channel is free/released and is prepared/finished for the acceptance of the following claim. The random character of the flow of claims leads to the fact that into some time intervals at the entrance of SMO accumulates the excessively large number of claims (they either form turn or they leave SMO not serviced); into other periods of SMO it will work with underloading or generally stay.

Each system of mass maintenance, depending on the number of channels and their productivity, and also on the character of the flow of claims, possesses some capacity, which allows for it more or less successfully to manage the flow of claims. Object/subject of queueing theory - establishment of the dependence between the character of the flow of claims, the number of channels, their productivity, the rules of the work of SMO and the success (efficiency) of maintenance.

As the characteristics of the efficiency of maintenance, depending on the conditions of problem and target/purposes of research, can be applied different values and the functions, for example:

- average quantitity of claims which can service SMO per unit time;

- the average percentage of claims, obtaining refusal and leaving SMO not serviced;

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- probability that acted claim will be immediately accepted for maintenance;

- mean latency in turn;

- the law of time allocation of expectation;

- an average quantity of claims, which are located in turn;

- the law of number distribution of claims in turn;

half speed, yielded SMO per unit time, etc.

The random character of the flow of claims, and in the general case and the duration of maintenance leads to the fact that in the system of mass maintenance will occur some random process. In order to give recommendation regarding the rational organization of this

process and to present the reasonable requirements for SMO, it is necessary to study the random process, which takes place in system, to describe it mathematically. By this is occupied queueing theory.

Let us note that in recent years the field of application of mathematical methods of queueing theory continuously is widened and everything more exceeds the limits of the problems, connected with the "operating organizations" literally. As the peculiar systems of mass maintenance can be examined: electronic digital computational machines; the system of collection and information processing; the automated production shops, assembly lines; transport systems; the air defense system, etc.

Close to the problems of queueing theory are many problems, which appear during the analysis of the reliability of technical equipment/devices.

The mathematical analysis of the work of SMO very is facilitated, if the random process, which takes place in system, is Markov. Then to it is possible comparatively simply describe work SMO with the help of the vehicle of ordinary differential (in the extreme case - linear algebraic) equations and to express in an explicit form the fundamental characteristics of the efficiency of maintenance through the parameters of SMO and flow of claims.

We know that, so that the process, which takes place in system, it would be Markov, it is necessary that all flows of events, translating system from state into state, they were Poisson (by flows without aftereffect). For SMO the flows of events - these are the flows of claims, the flows of the "maintenance" of claims, etc. If these flows are not Poisson, the mathematical description of the processes, which occur into SMO, it becomes incomparably more complex and requires the bulkier vehicle whose finishing/bringing to the explicit, analytical formulas is possible only in the rare, simplest cases. However, all the same vehicle of "Markov" queueing theory can prove useful, also, when the process, which takes place into SMO, is different from the Markov - with its aid of the characteristic of the efficiency of SMO they can be estimated approximately. It must be noted that than it is more complex SMO, than more in it the channels of maintenance, the fact more precise prove to be the approximation formulas, obtained with the help of Markov theory.

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One should also note that in a series of the cases for making of the substantiated decisions by control of the work of SMO completely and is not required precise knowledge of all its characteristics - often

sufficiently and approximated, tentative.

In present chapter will be presented the cell/elements of queueing theory, mainly in that simplest form which they acquire within the framework of Markov theory. For more detailed familiarization with queueing theory into its contemporary, developed form, the reader can be converted to the special monographs, for example, [14], [12], [20].

2. Classification of the systems of mass maintenance and their fundamental characteristics.

The systems of mass maintenance generally can be two types.

1. system with failures. In such systems the claim, which acted the torque/moment when all channels are occupied, obtains "failure", it leaves SMO and in further process of maintenance it does not participate.

2. Systems with expectation (with turn). In such systems the claim, which acted the torque/moment when all channels are occupied, stops in turn and expects until is freed one of the channels. As soon as will be freed channel, is accepted to maintenance one of the claims, which stand in turn.

Maintenance in system with expectation can be "that regulated" (claims are service/maintained by way of admission) and "not ordered" (claims are service/maintained in random order). Furthermore, in some SMO is applied the so-called "maintenance with priority", when some claims are service/maintained into first place, preferably before others.

Systems with turn are divided into systems with the unlimited expectation and systems with the limited expectation.

In systems with the unlimited expectation each claim, which acted the torque/moment when there are no free channels, stops in turn and "it pattently" awaits the release of the channel which will take it to maintenance. Any claim, which acted SMO, sooner or later will be serviced.

In systems with the limited expectation to the stay of claim in turn are superimposed one or the other limitations. These limitations can concern the length of turn (number of claims, which are simultaneously located in turn), of the retention time of claim in turn (after some period of the stay in turn claim leaves turn and it goes away), the total retention time of claim into SMO, etc.

In dependence from the type of SMO, during the evaluation of its efficiency can be applied one or the other values (indices of efficiency). For example, for SMO with failures of one of the most important characteristics of its productivity is the so-called absolute capacity, the mean number of claims,, which can be serviced by system for time unit.

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together with absolute, frequently is examined relative capacity SMO - average portion/fraction of the acted claims, operated by system (ratio of the average number of claims, operated system per unit time, to the average number of entering for this time claims).

Besides absolute and relative capacities, during the analysis of SMO with failures us they can, depending on the problem of research, interest other characteristics, for example:

- average number of occupied channels,

- a mean relative shutoff period of system as a whole and of separate channel, etc.

Let us pass to the examination of characteristics SMO with expectation.

Obviously, for SMO with the unlimited expectation both absolute and relative capacity they become meaningless, since each acted claim sooner or later will be serviced. Then for this SMO very important characteristics they are:

- Mean number of claims in turn,

- an average number of claims in system (in turn and under servicing),

- mean latency of claim in turn,

- mean retention time of claim in system (in turn and under maintenance),

and other characteristics of expectation.

For SMO with the limited expectation interest represent both groups of the characteristics: both the absolute and relative

capacity and the characteristic of expectation.

For the analysis of process, which takes place into SMO, is substantial to know the basic parameters of the system: the number of channels n, the intensity of flow of claims λ, the productivity of each channel (average number of claims μ, operated channel per unit time), of the condition of queueing (limitation, if they there is).

In dependence on these parameters we will subsequently express the characteristics of the efficiency of the work of SMO.

Previously let us agree (in order not to specify this is every time separately), that we will consider all flows of events, which translate SMO from state with state, Poisson. In those rare cases when we are examine not- Markov systems of mass maintenance, we will each time specify this specially.

Recall that in the case when Poisson flow is stationary (simplest flow), the interval of time T between events in this flow there is random variable, distributed according to the exponential law:

$$f(t)=\lambda e^{-\lambda t} \quad (t>0), \tag{2.1}$$

where λ - an intensity of flow of events.

In the case when from some state S_i system they derive/conclude immediately several of the simplest flows, value T - the retention time of system (in a row) in this state there is random variable, distributed according to the law (2.1), where λ - the total intensity of all flows of the events, which derive/conclude system from this state.

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3. Single-channel SMO with failures.

Let us consider protozoan of all problems of queueing theory - problem of the functioning of the single-channel of SMO with failures.

Let the system of mass maintenance consist only of one channel (n = 1) and it enters Poisson flow of claims with the intensity λ, which depends, in the general case, on the time:

$$\lambda = \lambda(t). \quad (3.1)$$

The claim, which found channel that occupied, obtains failure and leaves system.

The maintenance of claim is continued during random time distributed according to exponential law with the parameter μ:

$$f(t)=\mu e^{-\mu t} \quad (t>0). \qquad (3.2)$$

Hence it follows that the "flow of maintenance" - simplest, with intensity m. In order to visualize actually this flow, is imagined one continuously occupied channel - it will issue the serviced claims by flow with intensity μ.

Is required to find:

1) absolute capacity SMO (A);

2) relative capacity SMO (q).

Let us consider single channel of maintenance as physical system S, which can be located in one of the two states:

S_0 - free,

S_1 - occupied.

The graph/count of the states of system is shown on Fig. 5.1.

From state S_0 into S_1 the system, obviously, translates the flow of claims with intensity λ; S_1 and S_0 - the "flow of maintenance" with intensity μ.

Let us designate the probabilities of states $p_0(t)$ and $p_1(t)$. It is obvious, for any moment t

$$p_0(t)+p_1(t)=1. \tag{3.3}$$

Let us comprise differential of Kolmogorov's equations for the probabilities of states according to the rule, given in §3 chapters 4. We have:

$$\left.\begin{aligned}\frac{dp_0}{dt}&=-\lambda p_0+\mu p_1,\\ \frac{dp_1}{dt}&=-\mu p_1+\lambda p_0.\end{aligned}\right\} \tag{3.4}$$

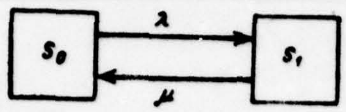

Fig. 5.1.

Of two equations (3.4) one it is excess, since p_0 and p_1 are connected by relationship/ratio (3.3). Taking into account this, let us reject/throw the second equation, and into the first let us substitute for p_1 its expression $(1-p_0)$:

$$\frac{dp_0}{dt} = -\lambda p_0 + \mu(1-p_0).$$

or

$$\frac{dp_0}{dt} = -(\mu+\lambda)p_0 + \mu. \qquad (3.5)$$

This equation logical to solve under the initial conditions:

$$p_0(0) = 1, \ p_1(0) = 0$$

(at the initial moment channel is free).

Linear differential equation (3.5) with one unknown function p_0 easily can be solved not only for the simplest flow of claims (λ = const), but also for case, when the intensity of this flow in the course of time changes ($\lambda = \lambda(t)$). Without being stopped on the last/latter case, let us give the solution of equation (3.5) only for

the case λ = const:

$$p_0 = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t}. \quad (3.6)$$

the dependence of value p_0 from time takes the form, depicted in Fig. 5.2. In initial moment (with t = 0) channel is knowingly free ($p_0(0)$ = 1). With an increase t probability p_0 is reduced and within limit (with $t \to \infty$) is equal to $\mu/(\lambda + \mu)$. Value $p_1(t)$, that supplements $p_0(t)$ to unity, is changed as shown in the same Fig. 5.2.

It is not difficult to ascertain that for single-channel SMO with failures probability p_0 is nothing else but the relative capacity q.

It is real/actual, p_0 there is probability that at torque/moment t the channel is free, otherwise probability that the claim, which came at torque/moment t, will be serviced. But that means that for the given torque/moment of time t, the average ratio of the number of serviced claims to the number of those acted also is equal p_0: $q = p_0$.

In limit, with $t \to \infty$, when the process of maintenance has already been establish/installed that the limiting value of relative capacity will be equal to:

$$q = \frac{\mu}{\lambda+\mu}. \quad (3.7)$$

Knowing the relative capacity q, it is easy to find absolute A. They are connected by the obvious relationship/ratio:

$$A = \lambda q. \tag{3.8}$$

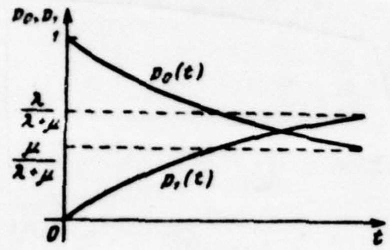

Fig. 5.2.

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In limit, with t -> ∞, absolute capacity also will be establish/installed and it will be equal to

$$A=\frac{\lambda\mu}{\lambda+\mu}. \qquad (3.9)$$

Knowing the relative capacity of system q (probability that come at torque/moment t claim will be serviced), it is easy to find failure probability:

$$P_{\text{отк}}=1-q. \qquad (3.10)$$

Failure probability $P_{\text{отк}}$ is nothing else but the mean portion/fraction of the unserviced claims among subjects. Within limit, with t -> ∞,

$$P_{\text{отк}}=1-\frac{\mu}{\lambda+\mu}=\frac{\lambda}{\lambda+\mu}. \qquad (3.11)$$

Example. Single-channel SMO with failures represents by itself one telephone line. Claim - call, which came at the torque/moment when line is occupied, obtains failure. Intensity of flow of calls λ = 0.8 (calls per minute). Average duration of conversation $\bar{t}_{об} = 1,5_{мин}$. All flows of events - simplest. To determine the limiting (with $t \to \infty$), value:

1) the relative capacity q;

2) the absolute capacity A;

3) failure probability $P_{отк}$.

To compare actual capacity SMO from nominal which will be, if each conversation lasts in accuracy 1.5 min, and conversations follow one another without interruption.

Solution. We determine the parameter μ of the flow of the maintenance:

$$\mu = 1/\bar{t}_{об} = 1/1,5 = 0,667.$$

On formula (3.6) we obtain relative capacity SMO:

$$q = \frac{0,667}{0,8 + 0,667} = 0,455.$$

Thus, in the steady-state conditicns/mode system will service/maintain about 45o/o of entering calls.

Through formula (3.9) we find the absclute capacity:

$$A=\lambda q=0{,}8\cdot 0{,}455=0{,}364,$$

i.e. line capable of carrying out on the average of 0.364 conversations per minute.

Failure probability:

$$P_{отк}=1-q=0{,}545,$$

means about 55o/o of acted calls will obtain failure.

Nominal spacing channel capacity:

$$A_{ном}=\frac{1}{\bar{t}_{об}}=0{,}667\ \text{(разговора в минуту)}.$$

Key: (1). (conversation per minute)

that almost is twice as more than the actual capacity, obtained taking into account the random character of the flow of claims and chance of the time of servicing.

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4. Multichannel SMO with failures.

Let us consider N-channel SMO with failures. Let us label the states of system according to the number of occupied channels (or, that in this case the same, according to the number of claims, connected with system). States will be:

S_0 - all channels are free,

S_2 - to occupy exactly one channel, remaining freedoms,

....

S_k - occupied exactly k of channels, the others are free,

....

S_n - occupied all n of channels.

Graph/count of the states SMO os represented in Fig. 5.3. Let us label graph/count, i.e., let us write at the arrow/pointers of the intensity of the corresponding flows of events. On arrow/pointers from left to right system transfers one and the same flow - flow of claims with intensity λ. If system is in state S_k (occupied k of

channels) arrived new claim system passes (it jumps) into state S_{k+1}.

Let us determine the intensities of flow of the events, which translate system according to arrow/pointers from right to left.

Let the system is be in state S_1 (is occupied one channel). Then, as soon as is finished the maintenance of the claim, which occupies this channel, system will pass in S_0; that means the flow of events, which translates system on arrow/pointer $S_1 \to S_0$, has intensity μ. It is obvious, if with the maintenance it is occupied two channels, and not one, flow of maintenance, which translates system on arrow/pointer $S_2 \to S_1$, will be double more intense (2μ); if is occupied k of channels - in k of times more intense ($k\mu$). Let us write the appropriate intensities of the arrow/pointers, which lead from right to left.

Figures 5.3 shows that the process, which takes place into SMO, represents by itself a special case of the process of death and multiplication, examined by us in § 8 chapters 4.

Using general rules, it is possible to comprise the equations of Kolmogorov for the probabilities of the states:

$$\left.\begin{aligned}
\frac{dp_0}{dt} &= -\lambda p_0 + \mu p_1,\\
\frac{dp_1}{dt} &= -(\lambda + \mu)\, p_1 + \lambda_1 p_0 + 2\mu p_1,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\\
\frac{dp_k}{dt} &= -(\lambda + k\mu)\, p_k + \lambda p_{k-1} + (k+1)\,\mu p_{k+1},\\
&\dots\dots\dots\dots\dots\dots\dots\dots\\
\frac{dp_n}{dt} &= -n\mu p_n + \lambda p_{n-1}.
\end{aligned}\right\} \qquad (4.1)$$

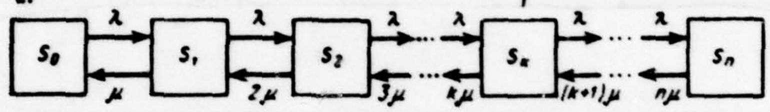

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Fig. 5.3.

Equations (4.1) they are called the equations of Erlang. Natural initial conditions for their solution they are:

$$p_0(0) = 1; \quad p_1(0) = p_2(0) = \dots = p_n(0) = 0$$

(at the initial moment system it is free).

The integration of system of equations (4.1) in analytical form is sufficiently complicated; in practice such systems of differential equations are usually solved numerically, by AVM or ETsVM. This solution gives to us all probabilities the states

$$p_0(t), \quad p_1(t), \dots, p_k(t), \dots, p_n(t)$$

as of function of time.

It is logical, us most of all they will interest the maximum

probabilities of states $p_0, p_1, \ldots, p_k, \ldots, p_n$, which characterize steady-load condition of SMC (with $t \to \infty$). For the determination of maximum probabilities, we will use already the prepared/finished solution of problem, obtained for the circuit of death and multiplication in § 8 chapters 4. According to this solution,

$$\left. \begin{aligned} p_k &= \frac{\lambda^k}{\mu \cdot 2\mu \ldots k\mu} p_0 = \frac{(\lambda/\mu)^k}{k!} p_0, \quad (k = 1, 2, \ldots, n); \\ p_0 &= \frac{1}{1 + \frac{\lambda/\mu}{1!} + \frac{(\lambda/\mu)^2}{2!} + \ldots \frac{(\lambda/\mu)^n}{n!}}. \end{aligned} \right\} \quad (4.2)$$

In these formulas the intensity of flow of claims λ and the intensity of flow of maintenance (for one channel) μ do not figure separately, but they enter only by their ratio λ/μ. Let us designate this sense

$$\lambda/\mu = \rho$$

and let us call value ρ the "given intensity" of the flow of claims. Its physical sense is such: value ρ represents by itself the average number of claims, which come in into SMC for the mean servicing time of one claim.

Taking into account this designation, formula (4.2) they take the form:

$$\left. \begin{aligned} p_k &= \frac{\rho^k}{k!} p_0, \quad (k = 1, 2, \ldots, n); \\ p_0 &= \frac{1}{1 + \frac{\rho}{1!} + \frac{\rho^2}{2!} + \ldots + \frac{\rho^n}{n!}} = \left[1 + \frac{\rho}{1!} + \frac{\rho^2}{2!} + \ldots + \frac{\rho^n}{n!} \right]^{-1}. \end{aligned} \right\} \quad (4.3)$$

Formulas (4.3) are called Erlang's formulas. They express the maximum probabilities of all states of system depending on the parameters λ, μ and n (λ - an intensity of flow of claims, μ - intensity of maintenance, n - a number of channels SMO).

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Knowing all probabilities of the states

$$p_0, p_1, \ldots, p_k, \ldots, p_n,$$

it is possible to find the characteristics of the efficiency of SMO: the relative capacity q, the absolute capacity A and failure probability $P_{\text{отк}}$.

It is real/actual, claim obtains failure, if it comes at the torque/moment when all n of channels are occupied. The probability of this is equal to

$$P_{\text{отк}} = p_n = \frac{\rho^n}{n!} p_0. \qquad (4.4)$$

Probability that claim will be accepted for maintenance (it the relative capacity q) it supplements $P_{отк}$ to unity:

$$q = 1 - p_n. \qquad (4.5)$$

Absolute capacity:

$$A = \lambda q = \lambda (1 - p_n). \qquad (4.6)$$

One Of the important characteristics SMO with failures is the average number of occupied channels (in this case it coincides with the average number of claims, which are located in system). Let us designate this average number $\bar{k}$.

Value $\bar{k}$ can be computed directly through probabilities $p_0, p_1, \ldots, p_n$ according to the formula:

$$\bar{k} = 0 \cdot p_0 + 1 \cdot p_1 + \ldots + n \cdot p_n \qquad (4.7)$$

as mathematical expectation of discrete random value which takes values of 0, 1, ..., n with probabilities $p_0, p_1, \ldots, p_n$. However, to considerably simply express the average number of occupied channels through the absolute capacity A, which we already know. It is real/actual, A is nothing else but the average number of claims, operated per unit time; one occupied channel it service/maintains on the average for the time unit μ of claims; the average number of occupied channels will be obtained by division A on μ:

$$\bar{k} = \frac{A}{\mu} = \frac{\lambda (1 - p_n)}{\mu}.$$

or, passing to designation $\lambda\mu - \rho$,

$$\bar{k} = \rho(1 - p_n). \quad (4.8)$$

Example. Are repeated the conditions of an example of the previous paragraph (λ = 0.8, μ = 0.667); however, instead of the single-channel SMO (n = 1) is examined three-channel (n = 3), i.e., the number of communication lines is increased to three. To find probability of failure and the average number of occupied channels.

Solution. Given intensity of flow of the claims:

$$\rho = \lambda/\mu = 0{,}8/0{,}667 = 1{,}2.$$

On Erlang's formulas (4.3) we obtain:

$$p_1 = \frac{\rho}{1!} p_0 = 1{,}2\, p_0,$$
$$p_2 = \frac{\rho^2}{2!} p_0 = 0{,}72\, p_0,$$
$$p_3 = \frac{\rho^3}{3!} p_0 = 0{,}288\, p_0,$$
$$p_0 = \frac{1}{1 + 1{,}2 + 0{,}72 + 0{,}288} \approx 0{,}312;$$
$$p_1 \approx 1{,}2 \cdot 0{,}312 \approx 0{,}374; \quad p_2 \approx 0{,}72 \cdot 0{,}312 \approx 0{,}224;$$
$$p_3 \approx 0{,}288 \cdot 0{,}312 \approx 0{,}090.$$

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We compute failure probability:

$$P_{отк} = p_3 = 0,090.$$

Relative and absolute capacities are equal to:

$$q = 1 - p_3 = 0,910; \quad A = \lambda q = 0,8 \cdot 0,910 = 0,728.$$

The average number of occupied channels:

$$\bar{k} = \rho(1 - p_3) = 1,2 \cdot 0,91 \approx 1,09,$$

i.e. during steady-load condition of SMC on the average will be occupied one with small channel of three - remaining two will stay. By this value is obtained a comparatively high level of the efficiency of maintenance - about 91o/o of all acted calls will be serviced.

5. Single-channel SMO with expectation.

Let us consider first protozoan of all possible SMO with expectation - single-channel system (n = 1), which enters the flow of claims with intensity λ; the intensity of maintenance μ (i.e. on the average the continuously occupied channel will issue μ the serviced claims in unity (time). The claim, which acted the torque/moment when

channel is occupied, stops into turn and expects maintenance.

Let us suppose that first that a quantity of places in turn is limited by Mach number, i.e., if claim arrived at torque/moment, when in turn already stand m of claims, it leaves system not serviced. In the future, after directing m to infinity, we will obtain the characteristics of the single-channel of SMO without limitations along the length of turn.

Let us label the states of SMO according to the number of claims, which are located in system (both operated and expecting maintenance):

S_0 - channel is free,

S_1 - a channel is occupied, there is no line,

S_2 - channel it is occupied, one claim stands in turn,

S_k - channel is occupied, k-1 claims stands in turn,

S_{m+1} - channel is occupied, m of claims stand in turn.

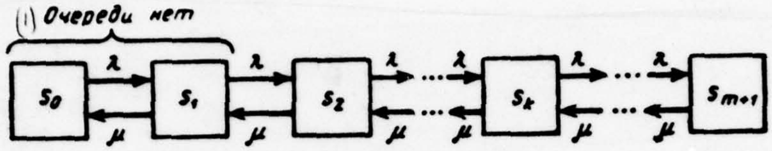

Fig. 5.4.
Key: (1). No turn.

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The graph/count of states SMO is shown on Fig. 5.4. The intensities of flow of the events, which translate into system on arrow/pointers from left to right, continually are equal to λ, and from right to left - μ. It is real/actual, on arrow/pointers from left to right the system translates the flow of claims (as soon as it will arrive claim, system it passes into the following state), to the right to the left - the flow of the "releases" of the occupied channel, which has intensity μ (as soon as it will be serviced next claim, channel either it will be freed or it decreases the number of claims in turn).

The depicted in Fig. 5.4 circuit represents by itself the circuit of death and multiplication. Using the general solution, given for the circuit of death and multiplication in §8 chapters 4, let us write the expressions of the maximum probabilities of the states:

$$\left.\begin{aligned} p_1 &= (\lambda/\mu)\, p_0, \\ p_2 &= (\lambda/\mu)^2\, p_0, \\ &\cdots\cdots\cdots\cdots \\ p_k &= (\lambda/\mu)^k\, p_0, \\ &\cdots\cdots\cdots\cdots \\ p_{m+1} &= (\lambda/\mu)^{m+1}\, p_0, \\ p_0 &= \frac{1}{1+(\lambda/\mu)+(\lambda/\mu)^2+\ldots+(\lambda/\mu)^{m+1}}. \end{aligned}\right\} \qquad (5.1)$$

Introducing designation $\lambda/\mu = \rho$, let us rewrite formulas (5.1) in the form:

$$\left.\begin{aligned}
&p_1 = \rho p_0,\\
&p_2 = \rho^2 p_0,\\
&\dots\dots\dots\dots\\
&p_k = \rho^k p_0,\\
&\dots\dots\dots\dots\\
&p_{m+1} = \rho^{m+1} p_0,\\
&p_0 = \frac{1}{1+\rho+\rho^2+\dots+\rho^{m+1}} =\\
&\quad = [1+\rho+\rho^2+\dots+\rho^{m+1}]^{-1}.
\end{aligned}\right\} \tag{5.2}$$

Let us note that in the denominator of last/latter formula (5.2) stands the geometric progression with first term 1 and denominator ρ; summarizing this progression, we find:

$$p_0 = \frac{1}{(1-\rho^{m+2})/(1-\rho)} = \frac{1-\rho}{1-\rho^{m+2}}. \tag{5.3}$$

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Thus, formulas (5.2) finally take the form:

$$\left.\begin{array}{l} p_0 = \frac{1-\rho}{1-\rho^{m+2}}, \\ p_1 = \rho p_0, \\ p_2 = \rho^2 p_0, \\ \cdots\cdots\cdots \\ p_k = \rho^k p_0, \\ \cdots\cdots\cdots \\ p_{m+1} = \rho^{m+1} p_0. \end{array}\right\} \qquad (5.4)$$

Let us focus attention on the fact that formula (5.3) is valid only when $\rho \neq 1$ (when $\rho = 1$ it gives the indeterminancy/uncertainty of form 0/0). But the sum of geometric progression with denominator $\rho = 1$ to find it is still simpler than on formula (5.3): it is equal $m + 2$, and in this case $p_0 = 1/(m + 2)$. Let us note that the same result we could obtain by more complex method, revealing indeterminancy/uncertainty (5.3) according to l'Hopital's rule.

Let us determine the characteristics of SMO: failure probability $P_{отк}$, the relative capacity q, the absolute capacity A, the average length of turn $\bar{r}$, the average number of claims, connected with system $\bar{k}$.

It is obvious, claim obtains failure only in the case when channel is occupied and all m of the places in turn - also:

$$P_{\text{отк}} = p_{m+1} = \frac{\rho^{m+1}(1-\rho)}{1-\rho^{m+2}}. \quad (5.5)$$

We find the relative capacity:

$$q = 1 - P_{\text{отк}} = 1 - \frac{\rho^{m+1}(1-\rho)}{1-\rho^{m+2}}. \quad (5.6)$$

Absolute capacity:

$$A = \lambda q.$$

Let us find the average number $\bar{r}$ of those locating in turn; let us define this value as mathematical expectation of discrete random variable R - number of claims, which are located in the turn:

$$\bar{r} = M[R].$$

With probability p_2 in turn stands one claim, with probability p_3 two claims, generally with probability p_k in turn stand k - 1 claims finally with probability p_{m+1} in turn stand m of claims. The average number of claims in turn we will obtain, multiplying the number of claims in turn by the appropriate probability and store/adding up the results:

$$\bar{r}=1\cdot p_2+2\cdot p_3+\ldots+(k-1)\cdot p_k+\ldots+m\cdot p_{m+1}=$$
$$=1\cdot\rho^2 p_0+2\cdot\rho^3 p_0+\ldots+(k-1)\cdot\rho^k p_0+\ldots+m\cdot\rho^{m+1}p_0. \quad (5.7)$$

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Let us take out in this expression $\rho^2 p_0$ for the brackets:

$$\bar{r}=\rho^2 p_0[1+2\rho+\ldots+(k-1)\rho^{k-2}+\ldots+m\rho^{m-1}]. \quad (5.8)$$

Let us deduce formula for the sum, which stands in the brackets (this formula we will frequently use subsequently). It is obvious, the sum in question represents it represents by itself nothing else but derivative on ρ the sums

$$\Sigma=\rho+\rho^2+\ldots+\rho^{k-1}+\ldots+\rho^m,$$

and for this expression we can use the formula of the sum of the geometric progression:

$$\Sigma=\frac{\rho-\rho^{m+1}}{1-\rho}. \quad (5.9)$$

Let us differentiate (5.9) on ρ:

$$\Sigma_\rho=\frac{[1-(m+1)\rho^m](1-\rho)+(\rho-\rho^{m+1})}{(1-\rho)^2}=$$
$$=\frac{1-(m+1)\rho^m-\rho+(m+1)\rho^{m+1}+\rho-\rho^{m+1}}{(1-\rho)^2}=$$
$$=\frac{1-(m+1)\rho^m+m\rho^{m+1}}{(1-\rho)^2}=\frac{1-\rho^m(m+1-m\rho)}{(1-\rho)^2}.$$

Thus, expression for the sum, which stands of brackets of right side (5.8), is found:

$$1+2\rho+\ldots+(k-1)\rho^{k-2}+\ldots+m\rho^{m-1}=\frac{1-\rho^m(m+1-m\rho)}{(1-\rho)^2}. \quad (5.10)$$

Substituting it in (5.8), we will obtain:

$$\bar{r}=\rho^2 p_0 \frac{1-\rho^m(m+1-m\rho)}{(1-\rho)^2}.$$

Taking into account expression for p_0 from (5.4), we have:

$$\bar{r}=\rho^2 \frac{(1-\rho)[1-\rho^m(m+1-m\rho)]}{(1-\rho^{m+2})(1-\rho)^2}$$

or, it is final,

$$\bar{r}=\frac{\rho^2[1-\rho^m(m+1-m\rho)]}{(1-\rho^{m+2})(1-\rho)}. \tag{5.11}$$

Thus, we will deduce expression for the average number of claims, expecting maintenance in turn. Let us deduce now formula for the average number $\bar{k}$ of the claims, connected with system (both standing in the turn and locating under maintenance). let us solve problem as follows: let us consider the total number of claims K, connected with system as sum of two random variables: the number of claims, which stand of turn, and the number of claims, which are located under the maintenance:

$$K=R+\Omega,$$

where R - a number of claims in turn, Ω - a number of claims under maintenance.

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According to the theorem of the addition of the mathematical

expectations

$$\bar{k} = M[K] = M[R] + M[\Omega] = \bar{r} + \bar{\omega},$$

where $\bar{r}$ - average number of claims in turn, $\bar{\omega}$ - average number of claims under maintenance.

Value $\bar{r}$ - we recently will find; let us find value $\bar{\omega}$. Since channel of us one, then random variable Ω can take only two values: 0 or 1. Value of 0 it takes, if the channel is free the probability of this is equal to

$$p_0 = \frac{1-\rho}{1-\rho^{m+2}}.$$

Value of 1 it takes, if channel is occupied; the probability of this is equal to

$$1 - p_0 = \frac{\rho - \rho^{m+2}}{1-\rho^{m+2}}.$$

We hence find the mathematical expectation of the number of claims, which are located under the maintenance:

$$\bar{\omega} = 0 \cdot p_0 + 1 \cdot (1 - p_0) = \frac{\rho - \rho^{m+2}}{1-\rho^{m+2}}.$$

Thus, the average number of claims, connected with SMO, will be

$$\bar{k} = \bar{r} + \frac{\rho - \rho^{m+2}}{1-\rho^{m+2}}, \qquad (5.12)$$

where value $\bar{r}$ is determined from formula (5.11).

Let us deduce expression for one more essential characteristic of SMO with the expectation: mean latency of claim in turn. Let us designate it $\bar{t}_{ож}$. Let the claim pass into system at some moment of

time. With probability p_0 the channel of maintenance will not be occupied, and by it it is not necessary to stand in the turn (latency is equal to zero). With probability p_1 it will arrive into system during the maintenance of some claim, but before it there will not be turns, and claim will await the beginning of its maintenance for a period of time $1/\mu$ (mean servicing time of one claim). With probability p_2 in the turn before the claim in question will stand one additional, and latency on the average will be equal $2/\mu$, and so forth. Generally, with probability p_k the come claim will find in system k of claims and it will await on the average k/μ unity of time; here k can be any integer to m.

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That concerns k = m + 1, i.e., the case when the newly incoming claim finds the channel of maintenance occupied and still m of claims in the turn (probability of this p_{m+1}), then latency in this case is also equal to zero, because claim does not get into line it is service/maintained). Therefore mean latency will be:

$$\bar{t}_{\text{ож}} = p_1 \frac{1}{\mu} + p_2 \frac{2}{\mu} + \ldots + p_k \frac{k}{\mu} + \ldots + p_m \frac{m}{\mu}.$$

Substituting here expressions for $p_1, \ldots, p_m$, we obtain:

$$\bar{t}_{\text{ож}} = p_0 \rho \frac{1}{\mu} + p_0 \rho^2 \frac{2}{\mu} + \ldots + p_0 \rho^k \frac{k}{\mu} + \ldots + p_0 \rho^m \frac{m}{\mu} = \frac{p_0 \rho}{\mu} (1 + 2\rho + \ldots + k\rho^{k-1} + \ldots + m\rho^{m-1}).$$

We convert the sum of brackets, using formula (5.10):

$$\bar{t}_{\text{ож}} = \frac{p_0 \rho}{\mu} \frac{1-\rho^m (m+1-m\rho)}{(1-\rho)^2},$$

or, expressing p_0 through ρ:

$$\bar{t}_{\text{ож}} = \frac{1}{\mu} \frac{\rho(1-\rho)}{1-\rho^{m+2}} \frac{1-\rho^m (m+1-m\rho)}{(1-\rho)^2} = \frac{\rho[1-\rho^m (m+1-m\rho)]}{\mu(1-\rho^{m+2})(1-\rho)}. \quad (5.13)$$

Equate/comparing this expression with formula (5.11), we note that

$$\bar{t}_{\text{ож}} = \frac{1}{\rho\mu} \bar{r} = \frac{\bar{r}}{\lambda}, \quad (5.14)$$

i.e. mean latency is equal to the average number of claims in turn, divided into the intensity of flow of claims.

Let us deduce another formula for the mean retention time of claim in system. Let us designate $T_{\text{сист}}$ random variable - retention time of claim into SMO. This random variable is composed of two terms (also random):

$$T_{\text{сист}} = T_{\text{ож}} + \Theta,$$

where $T_{\text{ож}}$ - latency of claim in turn, Θ - random variable, equal to servicing time $T_{\text{об}}$, if claim is service/maintained, and zero, if it is not service/maintained (obtains refusal).

According to the theorem of the addition of the mathematical expectations:

$$\bar{t}_{\text{сист}} = M[T_{\text{сист}}] = M[T_{\text{ож}}] + M[\Theta],$$

but, in our designations, $M[T_{\text{ож}}] = \bar{t}_{\text{ож}}$, and $M[\Theta] = q\bar{t}_{\text{об}} = q/\mu$. Hence let us

find: $\bar{t}_{\text{смот}} = \bar{t}_{\text{ож}} + q/\mu$, or, taking into account formula (5.4),

$$\bar{t}_{\text{смот}} = \bar{r}/\lambda + q/\mu. \quad (5.15)$$

Example 1. Autorefueling station (AZS) represents by itself SMO with one channel of maintenance (one column). Area/site with station allow/assumes the stay in turn for servicing not more than three machines simultaneously (m = 3).

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If in turn already is located three machines, the next machine, which arrived to station, in turn does not stop, but it will pass by. The flow of the machines, which arrive for servicing, has intensity $\lambda = 1$ (machine per minute). The process of servicing is continued in average/mean 1.25 min. To determine:

- failure probability.

- the relative and absolute capacity of SMO;

- average number of machines, which expect servicing.

- the average number of machines, which are found on AZS (including serviced).

- mean latency of machine in turn.

- the mean retention time of machine on AZS (including maintenance).

Solution. We find the given intensity of flow of the claims:

$$\mu = 1/1,25 = 0,8; \quad \rho = \lambda/\mu = 1/0,8 = 1,25.$$

On formulas (5.4):

$$p_0 = \frac{1 - 1,25}{1 - 3,05} \approx 0,122, \quad p_1 = 1,25 \cdot 0,122 \approx 0,152,$$
$$p_2 = 1,25 \cdot 0,122 \approx 0,191, \quad p_3 = 1,25 \cdot 0,122 \approx 0,238,$$
$$p_4 = 1,25 \cdot 0,122 \approx 0,297.$$

Failure probability $P_{отк} \approx 0,297$

Relative capacity SMC q = 1 - $P_{отк} = 0,703$.

Absolute capacity SMO A = λq = 0.703 (machine in min.).

The average number of machines in turn we find through formula (5.11)

$$\bar{r} = \frac{1,25^2 \, [1 - 1,25^3 \, (3 + 1 - 3,75)]}{(1 - 1,25^5)\,(1 - 1,25)} \approx 1,56.$$

i.e. the average number of machines, which expect in turn for servicing, is equal to 1.56.

Adjoining to this value the average number of machines, that are located under the maintenance

$$\bar{\omega} = \frac{1,25 - 1,25^5}{1 - 1,25^5} \approx 0,88,$$

we obtain the average number of machines, connected with AZS:

$$\bar{k} = \bar{r} + \bar{\omega} \approx 2,44.$$

Mean latency of machine in turn, on formula (5.14) is equal

$$\bar{t}_{ож} = \frac{\bar{r}}{\lambda} = 1,56 \text{ (min)}.$$

Adjoining to this value $M[\Theta] = \varrho/\mu = 0,703/0,8 \approx 0,88$, we will obtain the mean time which the machine carries out on AZS:

$$\bar{t}_{смот} = 1,56 + 0,88 = 2,44 \text{ (min)}.$$

Until now, we examine the work single-channel of SMO with expectation with the limited m number of the places in turn.

Let us now remove/take this limitation, i.e., let us direct m to infinity. In this case, the number of possible states of system will become infinite, and the graph/count of states will take the form, shown on Fig. 5.5.

Let us attempt to obtain the probabilities of states SMO with the unlimited turn by passage to the limit (with $m \rightarrow \infty$) from formulas (5.4).

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Let us note that in this case the denominator in last/latter formula (5.2) represents by itself the sum of the infinite number of terms of geometric progression. This sum descends only, when progression infinitely decreasing, i.e., when $\rho < 1$. It is possible completely strictly to demonstrate that $\rho < 1$ there is the condition, under which into SMO with expectation there is maximum being steady conditions/mode; when $\rho \geqslant 1$ such conditions/mode does not exist, and line with $t \rightarrow \infty$ increases to infinity.

Let us assume that

$$\rho = \lambda/\mu < 1,$$

i.e. that the maximum conditions/mode exists. Let us direct in formulas (5.4) m to ∞ and will deduce formulas for the maximum probabilities of states into SMO without limitations along the length of turn. We will obtain:

$$\left.\begin{aligned} p_0 &= 1-\rho, \\ p_1 &= \rho(1-\rho), \\ p_2 &= \rho^2(1-\rho), \\ &\dots\dots \\ p_k &= \rho^k(1-\rho), \\ &\dots\dots \end{aligned}\right| \qquad (5.16)$$

In the absence of limitations to to the length of turn each claim, which came into system, will be serviced; therefore q = 1, A = λq = λ. The average number of claims in turn we will obtain from (5.11) with m → ∞:

$$\bar{r} = \frac{\rho^2}{1-\rho}. \qquad (5.17)$$

The average number of claims in system on formula (5.12) with m → ∞ will be equal

$$\bar{k} = \bar{r} + \rho = \frac{\rho^2}{1-\rho} + \rho = \frac{\rho}{1-\rho}. \qquad (5.18)$$

Mean latency $\bar{t}_{ож}$ we will also obtain from formula (5.14) with m → ∞:

$$\bar{t}_{ож} = \frac{1}{\mu}\,\frac{\rho}{1-\rho}. \qquad (5.19)$$

or, in another form:

$$\bar{t}_{ож} = \frac{\rho^2}{\lambda(1-\rho)}. \qquad (5.20)$$

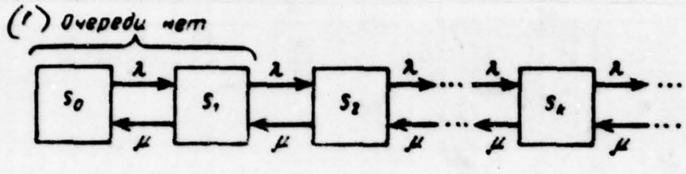

Fig. 5.5.

Key: (1). There is no turn.

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The mean retention time of claim into SMO is equal to mean latency plus mean servicing time $\bar{t}_{обсл} = 1/\mu$:

$$\bar{t}_{сист} = \frac{1}{\mu} \cdot \frac{\rho}{1-\rho} + \frac{1}{\mu} = \frac{1}{\mu} \cdot \frac{1}{1-\rho}. \quad (5.21)$$

Example of 2. To railroad hump yard arrive the compositions with intensity λ = 2 (composition in hour). The mean time, during which the hill processes composition, is equal to 0.4 hours. The compositions, which arrived at the torque/moment when hill is occupied, stop in turn and expect in the park/fleet of the arrival where there are three emergency route, for each of which can expect one composition the composition, which arrived at the torque/moment

when whole three emergency routes in the park/fleet of arrival are occupied, it stops in turn to external way. All flows of events - simplest. To find:

- the average number of compositions, which expect turn (both in the park/fleet of arrival and out of it).

- mean latency of composition in the park/fleet of arrival and on external ways.

- mean time of the determination of composition at sorting station (including expectation and maintenance).

- probability that the arrived composition will replace on external ways.

Solution. In our case $\lambda = 2$, $\mu = 1/0.4 = 2.5$, $\lambda/\mu = \rho = 2/2.5 = 0.8 < 1$, and SMO on the average "manages" the entering it flow of claims.

The average number of compositions, which expect turn (as in the park/fleet of arrival, so v out of it) let us find from formula (5.17):

$$\bar{r} = \frac{0,8^2}{1-0,8} = 3,2.$$

The average number of compositions, which expect turn on external ways, let us count as follows: with probability p_5 out of the park/fleet of arrival, will expect one composition, with probability p_6 - two compositions so forth, with probability $p_k(k > 5) - (k - 4)$ of composition.

The average number of compositions, which expect out of park/fleet, will be:

$$\bar{r}' = 1p_5 + 2p_6 + \ldots + (k-4)\, p_k + \ldots = 1\,(1-\rho)\,\rho^5 + 2\,(1-\rho)\,\rho^6 + \ldots +$$
$$+ \ldots + (k-4)\,(1-\rho)\,\rho^k + \ldots = (1-\rho)\,\rho^5\,[1 + 2\rho + 3\rho^2 + \ldots].$$

Formula for an infinite sum of brackets we obtain by passage to the limit (with $m \to \infty$) from formula (5.10):

$$1 + 2\rho + 3\rho^2 + \ldots = \frac{1}{(1-\rho)^2}. \qquad (5.22)$$

Hence

$$\bar{r}' = (1-\rho)\,\rho^5\,\frac{1}{(1-\rho)^2} = \frac{\rho^5}{1-\rho}.$$

Substituting here $\rho = 0.8$, we will obtain:

$$\bar{r}' = 1{,}64.$$

Probability that the arriving composition will replace on

external ways, is determined still simpler: it is equal to the probability of the fact that the length of turn will be not less than three, i.e.,

$$p_4+p_5+p_6+\dots=(1-\rho)\,\rho^4+(1-\rho)\,\rho^5+(1-\rho)\,\rho^6+\dots=$$
$$=(1-\rho)\,\rho^4\,(1+\rho+\rho^2+\dots)=\rho^4=0,8^4\approx 0,41.$$

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Mean latency in the park/fleet of arrival we determine, examining different hypotheses about the number of compositions, that are located in the system:

$$\frac{1}{\mu}p_1+\frac{2}{\mu}p_2+\frac{3}{\mu}[p_3+p_4+p_5+\dots]=$$
$$=\frac{1}{\mu}\{1\cdot\rho(1-\rho)+2\rho^2(1-\rho)+3[\rho^3(1-\rho)+\rho^4(1-\rho)+\dots]\}=$$
$$=\frac{1}{\mu}[\rho(1-\rho)+2\rho^2(1-\rho)+3\rho^3]=$$
$$=\frac{1}{\mu}[\rho-\rho^2+2\rho^2-2\rho^3+3\rho^3]=\frac{1}{\mu}[\rho+\rho^2+\rho^3]=\frac{1}{\mu}\,\frac{\rho-\rho^4}{1-\rho}.$$

For ρ = 0.8, μ = 2.5, we obtain, that mean latency in the park/fleet of arrival is equal

$$0,4\,\frac{0,8-0,41}{0,2}=0,78 \text{ (hour)} \approx 47 \text{ (min)}.$$

As far as latency concerns be concerned on external ways, it is equal

$$\frac{1}{\mu}p_4+\frac{2}{\mu}p_5+\frac{3}{\mu}p_6+\dots=\frac{1}{\mu}(1-\rho)\,\rho^4+\frac{2}{\mu}(1-\rho)\,\rho^5+\frac{3}{\mu}(1-\rho)\,\rho^6+\dots=$$
$$=\frac{(1-\rho)\,\rho^4}{\mu}[1+2\rho+3\rho^2+\dots]=\frac{(1-\rho)\,\rho^4}{\mu}\,\frac{1}{(1-\rho)^2}=\frac{1}{\mu}\,\frac{\rho^4}{1-\rho},$$

i.e. for our numerical data,

$$0{,}4\cdot 0{,}41/0{,}2=0{,}82 \text{ (hour)} \approx 49 \text{ (min)}.$$

The mean retention time of composition at sorting station (counting expectation and maintenance) will be equal to:

$$\bar{t}_{\text{смст}}=0{,}82+0{,}78+0{,}4=2 \text{ (hour)}.$$

6. Multichannel SMO with expectation.

Let us consider n-channel SMO with expectation, which enters the flow of claims with intensity λ; the intensity of maintenance (for one channel) μ; the number of places in turn m.

The states of system let us label according to the number of claims, connected with the system:

(1) Очереди нет:

S_0 — все (2) каналы свободны,
S_1 — занят один канал, остальные свободны,
. . (3) . . . . (4) . . . .
S_k — заняты k каналов, остальные свободны,
. . . . (5) . . . .
S_n — заняты все n каналов,
(6)
S_{n+1} — заняты все n каналов; одна заявка стоит в очереди,
. . . (7) . . . .
S_{n+r} — заняты все n каналов, r заявок стоят в очереди,
. . . (8) . . . .
S_{n+m} — заняты все n каналов, m заявок стоят в очереди.

Key: (1). There is no turn. (2). all channels are free. (3). is occupied one channel, others are free. (4). are occupied k of channels, others are free. (5). are occupied all n of channels. (6). occupying all n of channels; one claim is worth in turn. (7). Are busy all n of channels,... of claims stand in turn.

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The graph/count of states is given to Fig. 5.4. At each arrow/pointer are written the corresponding intensities of flow of events. It is real/actual, on arrow/pointers from left to right the system translates always one and the same flow of claims with intensity λ; on arrow/pointers from right to left the system translates the flow of maintenance whose intensity is equal to μ, multiplied by the number of occupied channels.

Graph/count in Fig. 5.6 represents by himself the circuit of death and multiplication, for which the solution in general form is already obtained. Let us write expressions for the maximum probabilities of states, immediately designating $\lambda/\mu = \rho$:

$$p_1 = \frac{\rho}{1!} p_0, \quad p_2 = \frac{\rho^2}{2!} p_0, \quad \ldots, \quad p_n = \frac{\rho^n}{n!} p_0,$$

$$p_{n+1} = \frac{\rho^{n+1}}{n n!} p_0, \quad p_{n+2} = \frac{\rho^{n+2}}{n^2 n!} p_0, \quad \ldots, \quad p_{n+m} = \frac{\rho^{n+m}}{n^m n!} p_0,$$

$$p_0 = \left[1 + \frac{\rho}{1!} + \frac{\rho^2}{2!} + \ldots + \frac{\rho^n}{n!} + \underline{\frac{\rho^{n+1}}{n n!} + \frac{\rho^{n+2}}{n^2 n!} + \ldots + \frac{\rho^{n+m}}{n^m n!}}\right]^{-1}$$

or, summarizing geometric progression with denominator ρ/n the emphasized terms):

$$\left.\begin{aligned}
p_0 &= \left[1+\frac{\rho}{1!}+\frac{\rho^2}{2!}+\ldots+\frac{\rho^n}{n!}\cdot\frac{\rho/n-(\rho/n)^{m+1}}{1-\rho/n}\right]^{-1},\\
p_1 &= \frac{\rho}{1!}\,p_0,\\
p_2 &= \frac{\rho^2}{2!}\,p_0,\\
&\cdots\cdots\cdots\cdots\\
p_n &= \frac{\rho^n}{n!}\,p_0,\\
p_{n+1} &= \frac{\rho^n}{n\cdot n!}\,p_0,\\
p_{n+2} &= \frac{\rho^{n+2}}{n^2\cdot n!}\,p_0,\\
&\cdots\cdots\cdots\cdots\\
p_{n+m} &= \frac{\rho^{n+m}}{n^m\cdot n!}\,p_0.
\end{aligned}\right\} \qquad (6.1)$$

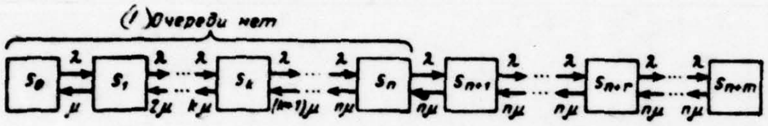

Fig. 5.6.

Key: (1). There is no turn.

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Thus, all probabilities of states are found.

Let us find some characteristics of the efficiency of servicing. The acted claim obtains failure, if are occupied all n of channels and all m of the places in the turn:

$$P_{\text{отк}} = p_{n+m} = \frac{\rho^{n+m}}{n^m \, n!} p_0. \qquad (6.2)$$

Relative capacity, as ever, supplements failure probability to unity:

$$q = 1 - P_{\text{отк}} = 1 - \frac{\rho^{n+m}}{n^m \cdot n!} p_0.$$

Absolute capacity SMO will be equal to:

$$A = \lambda q = \lambda \left(1 - \frac{\rho^{n+m}}{n^m \cdot n!} p_0\right). \qquad (6.3)$$

Let us find the average number of occupied channels. For SMO with failures, it coincides with the average number of claims, which were being located in system. For SMO with turn, the average number of occupied channels does not coincide with the average number of claims, which are located in the system: last/latter value differs from the first to average number of claim, in the queue. We retain from the designation $\bar{k}$ for the mean number of claims connected with system, but the average number of occupied channels let us designate $\bar{z}$. Each occupied channel it service/maintains on the average μ of claims per unit time; entire/all SMO service/maintains on the average A of claims per unit time. Dale one to another, we will obtain:

$$\bar{z}=\frac{A}{\mu}=\frac{\lambda}{\mu}\left(1-\frac{\rho^{n+m}}{n^m\cdot n!}p_0\right).$$

or

$$\bar{z}=\rho\left(1-\frac{\rho^{n+m}}{n^m\cdot n!}p_0\right). \qquad (6.4)$$

The average number of claims in turn can be computed directly as mathematical expectation of discrete random variable, multiplying any possible number of claims for probability that precisely this number of claims it will be in turn, and store/adding up the results:

$$\bar{r}=1\cdot p_{n+1}+2\cdot p_{n+2}+\ldots+m\cdot p_{n+m}=1\cdot\frac{\rho^{n+1}}{n\cdot n!}p_0+2\cdot\frac{\rho^{n+2}}{n^2\cdot n!}p_0+\ldots+$$
$$+m\cdot\frac{\rho^{n+m}}{n^m\cdot n!}p_0=\frac{\rho^{n+1}p_0}{n\cdot n!}\left[1+2\frac{\rho}{n}+3\left(\frac{\rho}{n}\right)^2+\ldots+m\left(\frac{\rho}{n}\right)^{m-1}\right]. \qquad (6.5)$$

Let us introduce designation $\rho/n=x$ and let us rewrite (6.5) in

the form:

$$\bar{r}=\frac{\rho^{n+1}}{n\cdot n!}p_0[1+2x+3x^2+\ldots+mx^{m-1}]. \quad (6.6)$$

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Let us note that the bracketed expression is nothing else but already calculated by us of the previous paragraph sum (5.10), where instead of ρ is placed x. Using this formula and substituting result in (6.6), we will obtain:

$$\bar{r}=\frac{\rho^{n+1}p_0}{n\cdot n!}\cdot\frac{1-(m+1)x+mx}{(1-x)^2}. \quad (6.7)$$

Store/adding up the average number of claims in turn $\bar{r}$ and the average number of occupied channels $\bar{z}$, we will obtain the average number of claims, connected with the system:

$$\bar{k}=\bar{z}+\bar{r}. \quad (6.8)$$

Let us now find mean latency of claim in the turn: $\bar{t}_{\text{ож}}$. Let us do a series of hypotheses about state in which will find system the newly come claim and how long by it it is necessary to await maintenance.

If claim finds not all channels occupied, by it not at all it is necessary to wait (corresponding terms in mathematical expectation let us reject/throw how equal to zero). If claim arrives at

torque/moment, when are cccupied all n cf channels, but there is no turn, by it it is necessary to await cr the average the time, equal to $1/n\mu$ (because the flow cf releases n of channels has intensity $n\mu$). If claim finds all channels occupied and one claim before itself in turn, by it it is necessary on the average tc await time $2/n\mu$ (on in turn r of claims, by it it is necessary to await on the average time $r/n\mu$. If the newly ccme claim finds in turn already m of claims, then it will not at all await (but alsc will nct be service/maintained). Mean latency let us find, multiplying each of these values for the appropriate probability:

$$\bar{t}_{\text{ож}} = \frac{1}{n\mu} p_n + \frac{2}{n\mu} p_{n+1} + \ldots + \frac{m}{n\mu} p_{n+m-1} =$$
$$= \frac{1}{n\mu}\left[\frac{\rho^n}{n!} p_0 + \frac{2\rho^{n+1}}{n \cdot n!} p_0 + \ldots + \frac{m\rho^{n+m-1}}{n^{m-1} \cdot n!} p_0\right] =$$
$$= \frac{\rho^n p_0}{n \cdot n! \, \mu}\left[1 + \frac{2\rho}{n} + \frac{3\rho^2}{n^2} + \ldots + \frac{m\rho^{m-1}}{n^{m-1}}\right].$$

Just as in the case of the single-charnel of SMO with expectation, we note that this expression differs from expression for the average length of turn (6.5) only in terms of factor $1/\rho\mu = 1/\lambda$, i.e.,

$$\bar{t}_{\text{ож}} = \frac{\bar{r}}{\lambda}. \tag{6.9}$$

Substituting here expression for $\bar{r}$, let us find:

$$\bar{t}_{\text{ож}} = \frac{\rho^n p_0}{n\mu n!} \frac{1-(m+1)\varkappa^m + m\varkappa^{m+1}}{(1-\varkappa)^2}. \tag{6.10}$$

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The mean retention time of claim in system, just as for single-channel SMO, differs from mean latency to the mean servicing time, multiplied to the relative capacity:

$$\bar{t}_{\text{сист}} = M[T_{\text{ож}}] + M[\Theta] = \bar{t}_{\text{ож}} + q/\mu. \qquad (6.11)$$

Example 1. Autoservicing station (AZS) with two columns (n = 2) is intended for maintain/serviceing the machines. The flow of the machines, which arrive on AZS, has intensity $\lambda = 2$ (machine per minute); the mean time of the tinning of one machine

$$\bar{t}_{\text{об}} = 1/\mu = 2 \text{ (min)}.$$

Area/site of AZS can contain turn not more than m = 3 (machines). The machine, which arrived at the torque/moment when all the three places in turn are occupied, leaves AZS (is obtained failure). To find the characteristics of SMO:

- failure probability.

- the relative and absolute capacity.

- the average number of occupied columns.

- the average number of machines in turn.

- mean latency and stay of machine on AZS.

Solution. We have: n = 2, m = 3, λ = 2, $\mu = 1/\bar{t}_{об} = 0,5$, ρ = 4, $\varkappa = \rho/n = 2$.

Through formulas (6.1) we find:

$$p_0 = \frac{1}{1 + \frac{4}{1} + \frac{4^2}{2} + \frac{4^2}{2}\,\frac{2-2^4}{1-2}} = \frac{1}{125} = 0,008.$$

Failure probability:

$$P_{отк} = p_{n+m} = p_5 = \frac{4^5}{2^3 \cdot 2}\,p_0 = 64 p_0 = 0,512.$$

Relative capacity:

$$q = 1 - P_{отк} = 0,488.$$

The absolute capacity:

A = qλ = 0.976 (machine per minute).

The average number of occupied channels (columns):

$$\bar{z} = A/\mu = 0,976/0,5 = 1,952$$

(i.e. both columns almost always are occupied).

The average number of machines in turn we find through formula (6.7):

$$\bar{r} = \frac{4^3}{2\cdot 2\cdot 125}\,\frac{1-4\cdot 2^3+3\cdot 2^4}{(1-2)^2} = \frac{16}{125}\,17 = 2,18$$

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Mean latency in turn - on formula (6.9):

$$\bar{t}_{\text{ож}} = \bar{r}/\lambda = 2,18/2 = 1,09 \text{ (min)}.$$

Mean retention time of machine on AZS (including the time of tinning):

$$\bar{t}_{\text{сист}} = \bar{t}_{\text{ож}} + q\bar{t}_{\text{об}} = 1,09 + 0,976 = 2,07 \text{ (min)}.$$

Above we considered n-channel SMO with the expectation when in turn simultaneously can be located not more than m claims.

Just as in the previous paragraph, let us look, which will be, if the length of turn is not limited by some Mach number, but there

can be how convenient large. The graph/count of states in this case - infinite (see Fig. 5.7).

The probabilities of states we will obtain from formulas (6.1) by passage to the limit (with m $\rightarrow$ ∞). Let us note that the sum of the corresponding geometric progression descends when $x = \rho/n < 1$ and it diverges when $x \geq 1$; respectively, steady mode will exist when $x < 1$, and when $x > 1$ turn will infinitely grow. Let us assume that $x < 1$ let us direct in formulas (6.1) value m to infinity. We will obtain expressions for the maximum probabilities of the states:

$$\left.\begin{aligned}
p_0 &= \left[1 + \frac{\rho}{1!} + \frac{\rho^2}{2!} + \ldots + \frac{\rho^n}{n!} + \frac{\rho^{n+1}}{n!\,(n-\rho)}\right]^{-1}, \\
p_1 &= \frac{\rho}{1!}\, p_0, \\
p_2 &= \frac{\rho^2}{2!}\, p_0, \\
p_n &= \frac{\rho}{n!}\, p_0, \\
p_{n+1} &= \frac{\rho^{n+1}}{n \cdot n!}\, p_0, \\
p_{n+2} &= \frac{\rho^{n+2}}{n^2 \cdot n!}\, p_0, \\
&\ldots\ldots\ldots\ldots \\
p_{n+r} &= \frac{\rho^{n+r}}{n^r \cdot n!}\, p_0, \\
&\ldots\ldots\ldots\ldots
\end{aligned}\right| \quad (6.12)$$

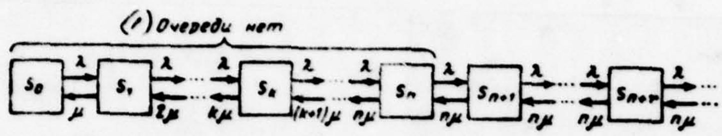

Fig. 5.7.

Key: (1). There is no turn.

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Since each claim sooner or later will be serviced, then the characteristics of the capacity of SMO are equal to

$$P_{отк} = 0, \quad q = 1, \quad A = \lambda q = \lambda.$$

The average number of claims in turn we will obtain with m → - from (6.7):

$$\bar{r} = \frac{\rho^{n+1} p_0}{n \cdot n! \, (1 - \varkappa)^2}, \tag{6.13}$$

while mean latency - from (6.10):

$$\bar{t}_{ож} = \frac{\rho^n p_0}{n \mu \, n! (1 - \varkappa)^2}. \tag{6.14}$$

The average number of occupied channels $\bar{z}$ will be located as before through the absolute capacity:

$$\bar{z} = \frac{A}{\mu} = \frac{\lambda}{\mu} = \rho, \tag{6.15}$$

while the average number of claims, connected with SMO - as average number of claims in turn plus the average number of claims, which are located under maintenance (average number of occupied channels):

$$\bar{k}=\bar{r}+\bar{z}. \quad (6.16)$$

Example 2. Autoservicing station with two columns (n = 2) service/maintains the flow of machines with intensity λ = 0.8 (machines per minute). Mean servicing time of one machine

$$\bar{t}_{об}=\frac{1}{\mu}=2 \text{ (min)}.$$

In this region no other of AZS, so that the turn of the machines before AZS can increase virtually unlimitedly. To find the characteristics of SMO.

Solution. We have: n = 2, λ = 0.8, $\mu=1/\bar{t}_{об}=0,5$, ρ = 1.6, $\varkappa=\rho/n=0.8$. **Since** $\varkappa<1$, the turn does not increase infinite and makes sense to speak about maximum steady-state operating conditions of SMO. On formulas (6.11) on is ongoing the probability of the states:

$$p_0=\left[1+1,6+1,28+\frac{4,09}{2\cdot 0,4}\right]^{-1}\approx 0,111,$$

$$p_1=1,6p_0\approx 0,178,\quad p_2=1,28p_0\approx 0,142,$$

$$p_3=\frac{1,6^3}{2\cdot 2!}p_0\approx 0,114,\quad p_4=\frac{1,6^4}{2^2\cdot 2!}p_0\approx 0,091$$ and i.e.

the average number of occupied channels let us find, after dividing absolute capacity SMO A = λ = 0.8 into the intensity of maintenance μ = 0.5:

$$\bar{z}=0,8/0,5=1,6.$$

The probability of the absence of turn of AZS will be:

$$p_0+p_1+p_2\approx 0,431.$$

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Average number of machines in the turn:

$$\bar{r}=\frac{1,6^3\cdot 0,111}{2\cdot 2\cdot 0,42^2}\approx 0,71.$$

Average number of machines on AZS:

$$\bar{k}=\bar{r}+\bar{z}\approx 0,71+1,6=2,31$$

Mean latency in the turn:

$$\bar{t}_{ож} = \frac{\bar{r}}{\lambda} \approx 0{,}89 \text{ (min)}.$$

Mean retention time of machine on AZS:

$$\bar{t}_{сист} = \bar{t}_{ож} + \bar{t}_{об} \approx 0{,}89 + 2 = 2{,}89 \text{ (min)}.$$

7. SMO With limited latency.

Until now, we examine SMO with the expectation, limited only by the length of turn (by Mach number of claims, which were being simultaneously located in turn). In such SMO claim, once been in turn, no longer leaves it and "pattently" it waits until maintenance. In practice frequently they are encountered SMO of another type, in which the claim, after waiting certain time, can go away from turn (the so-called "impatient" claims).

Let us consider similar type SMO, remaining within the framework of Markov circuit. Let us assume that there is N-channel SMO with expectation, in which the number of places in turn not is limited, but the retention time of claim in turn is limited by certain random period $T_{оч}$ with average value $\bar{t}_{оч}$, thus, to each claim, which stands

in turn, functions as the "flow of attendance/departures" with the intensity

$$\nu = \frac{1}{\bar{t}_{\text{оч}}}.$$

If this flow is Poisson, then the process, taking place in SMO, will be Markov. Let us find for it the probabilities of states.

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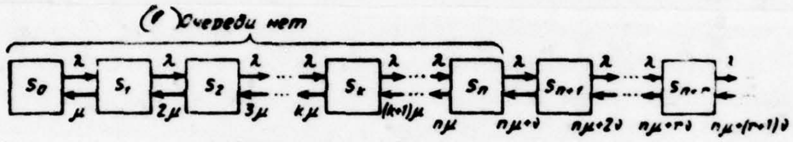

FIG. 5.8. Key: (1). there is no turn.

Let us again label the states of system according to the number of claims, connected with system - both operated and standing in the turn:

(1) очереди нет:
- S_0 — (2) все каналы свободны,
- S_1 — (3) занят один канал,
- S_2 — (4) заняты два канала,
- . . . .
- S_n — (5) заняты все n каналов,

S_{n+1} — (6) заняты все n каналов, одна заявка стоит в очереди,

. . . .

S_{n+r} — (7) заняты все n каналов, r заявок стоит в очереди,

. . . .

and so forth.

Key: (1). there is no turn. (2). all channels are free. (3). is occupied one channel. (4). are occupied two channels. (5). are occupied all n of channels. (6). are occupied all n of channels, one claim stands in turn. (7). are occupied all n of channels, r of claims stands in turn.

The graph/count of the states of system is shown on Fig. to 5.8. Let us label this graph/ccurt, i.e., let us write at arrow/pointers the appropriate intensities. Again, as before at all rifleman/gunner, that lead from left to right, it will stand the intensity of flcw of claims λ. For states without turn the arrow/pointers, which lead of them from right to left, will have, as before tc stand the total intensity of flow of maintenance all occupied of them will from right to left stand the total intensity of flcw of maintenance all n of channels $n\mu$, plus the corresponding intensity of flow of attendance/departures from turn. If in turn stand r of claims, then is the total intensity of flow of attendance/departures from turn. If in turn stand r of claims, then the total intensity of flow of attendance/departures will be equal to $r\nu$.

As can be seen from graph, before us again the circuit of death and multiplication, applying common/general/total expressions for the maximum probabilities of states in this circuit, let us write:

$$p_1 = \frac{\lambda/\mu}{1!} p_0,$$

$$p_2 = \frac{(\lambda/\mu)^2}{2!} p_0,$$

$$\cdots\cdots\cdots\cdots$$

$$p_n = \frac{(\lambda/\mu)^n}{n!} p_0,$$

$$p_{n+1} = \frac{(\lambda/\mu)^n \lambda}{n!\,(n\mu + \nu)} p_0,$$

$$p_{n+2} = \frac{(\lambda/\mu)^n}{n!} \frac{\lambda^2}{(n\mu + \nu)(n\mu + 2\nu)} p_0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$p_{n+r} = \frac{(\lambda/\mu)^n}{n!} \frac{\lambda^r}{(n\mu + \nu)(n\mu + 2\nu)\ldots(n\mu + r\nu)} p_0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$p_0 = \left\{1 + \frac{\lambda/\mu}{1!} + \frac{(\lambda/\mu)^2}{2!} + \ldots + \frac{(\lambda/\mu)^n}{n!} + \frac{(\lambda/\mu)^n}{n!}\left[\frac{\lambda}{n\mu + \nu} + \right.\right.$$

$$\left.\left. + \frac{\lambda^2}{(n\mu + \nu)(n\mu + 2\nu)} + \ldots + \frac{\lambda^r}{(n\mu + \nu)(n\mu + 2\nu)\ldots(n\mu + r\nu)} + \ldots\right]\right\}^{-1},$$

or, introducing the designations:

$$\rho = \lambda/\mu, \quad \beta = \nu/\mu,$$

$$\left.\begin{array}{l}
p_0 = \left\{1 + \frac{\rho}{1!} + \frac{\rho^2}{2!} + \ldots + \frac{\rho^n}{n!} + \frac{\rho^n}{n!}\left[\frac{\rho}{n+\beta} + \right.\right. \\
\left.\left. + \frac{\rho^2}{(n+\beta)(n+2\beta)} + \ldots + \frac{\rho^r}{(n+\beta)(n+2\beta)\ldots(n+r\beta)} + \ldots\right]\right\}^{-1}, \\
p_1 = \frac{\rho}{1!} p_0, \\
p_2 = \frac{\rho^2}{2!} p_0, \\
\cdots\cdots\cdots\cdots \\
p_n = \frac{\rho^n}{n!} p_0, \\
p_{n+1} = \frac{\rho^n}{n!} \frac{\rho}{n+\beta} p_0, \\
p_{n+2} = \frac{\rho^n}{n!} \frac{\rho^2}{(n+\beta)(n+2\beta)} p_0, \\
\cdots\cdots\cdots\cdots\cdots\cdots \\
p_{n+r} = \frac{\rho^n}{n!} \frac{\rho^r}{(n+\beta)(n+2\beta)\ldots(n+r\beta)} p_0, \\
\cdots\cdots\cdots\cdots
\end{array}\right\} \quad (7.1)$$

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Let us note some special feature/peculiarities by that examined of SMO o by "impatient" claims in comparison with by that previously examined SMO with "patient claims.

If the length of turn not it is limited previously by any number and claims "are patient" (they do not go away from turn), then steady maximum state exists only in the case $\rho < n$ (with $\rho \geq n$ the corresponding infinite geometric progression diverges, which physically corresponds to the unlimited increase in the turn in $t \to \infty$). On the contrary, into SMO with the "impatient" claims, which exit sooner or later from the turn, which was steady the conditions/mode of maintenance with $t \to \infty$ is reached always, independent of the given intensity of flow of claims ρ. This follows from the fact that a series in the denominator of first formula (7.1) descends at any positive values ρ and β.

For SMO with "impatient" claims, the concept "failure probability" does not have sense - each claim it stops in turn, but can and not wait until itself maintenance, leaving ahead of time.

The relative capacity q of such SMC can be counted as follows. It is obvious, are serviced will be all claims, except those that will go away from turn before the appointed time. Let us count which on the average number of claims it leaves turn before the appointed time. For this, let us compute the average number of claims in the turn:

$$\bar{r} = 1 \cdot p_{n+1} + 2 \cdot p_{n+2} + \ldots + r \cdot p_{n+r} + \ldots \quad (7.2)$$

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For each of these claims, functions the "flow of attendance/departures" with intensity ν. That means that from the average number $\bar{r}$ of claims in turn on the average, it will go away, ahaving waited until maintenance, $\nu\bar{r}$ claims per unit time; in all per unit time on the average it will be serviced

$$A = \lambda - \nu\bar{r} \quad (7.3)$$

claims. Relative capacity SMO will be

$$q = \frac{A}{\lambda} = \frac{\lambda - \nu\bar{r}}{\lambda} = 1 - \frac{\nu}{\lambda}\bar{r}. \quad (7.4)$$

The average number of occupied channels $\bar{z}$ we will obtain as before that Dale absolute capacity on μ:

$$\bar{z} = \frac{A}{\mu} = \frac{\lambda - \nu\bar{r}}{\mu} = \rho - \beta\bar{r}. \quad (7.5)$$

This makes it possible to compute the average number of claims

in turn $\bar{r}$, without summarizing infinite series (7.2). It is real/actual, from (7.5) we will obtain:

$$\bar{r}=\frac{\rho}{\beta}-\frac{\bar{z}}{\beta}, \quad (7.6)$$

and the entering this formula average number of occupied channels can be found as mathematical expectation of random variable Z, which takes values of 0, 1, 2, ..., n with probabilities p_0, p_1, p_2, ..., $[1-(p_0+p_1+...+p_{n-1})]$:

$$\bar{z}=0\cdot p_0+1\cdot p_1+2\cdot p_2+...+n\cdot[1-(p_0+p_1+...+p_{n-1})]= \\ =p_1+2p_2+...+n[1-(p_0+p_1+...+p_{n-1})]. \quad (7.7)$$

We will not derive formulas for mean latency in turn, since for this are required comparatively complex lining/calculations.

Let us note that, unlike formulas §§ 5, 6, where the sums of the large (or infinite) number of terms "are coagulated" with the help of formulas for the sum of geometric progression, in formula (7.1) figures the sum of the infirite series, which is not progression. However, this sum is computed approximately, moreover it is sufficiently easy, since the terms of a series rapidly decrease with an increase in their number. As approximate value for an infinite sum, is taken the sum of the finite number r-1 of terms, and residue/remainder is considered as follows:

$$\frac{\rho^n}{n!}\left[\frac{\rho^r}{(n+\beta)(n+2\beta)\dots(n+r\beta)}+\frac{\rho^{r+1}}{(n+\beta)(n+2\beta)\dots(n+(r+1)\beta)}+\dots\right]<$$
$$<\frac{\rho^n}{n!}\left[\frac{\rho^r}{\beta\, 2\cdot\beta\cdot\dots\cdot r\,\beta}+\frac{\rho^{r+1}}{\beta\cdot 2\,\beta\cdot\dots\cdot(r+1)\,\beta}+\dots\right]=$$
$$=\frac{\rho^n}{n!}\left[\frac{(\rho/\beta)^r}{r!}+\frac{(\rho/\beta)^{r+1}}{(r+1)!}+\dots\right]. \qquad (7.8)$$

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It is possible to **demonstrate** that the infinite sum of the brackets is less than $\frac{(\rho/\beta)^r}{r!}e^{\rho/\beta}$, and expression (7.8) is less than

$$\frac{\rho^n}{n!}\frac{(\rho/\beta)^r}{r!}e^{\rho/\beta}.$$

In conclusion let us note that if we f formulas (7.1) pass to limit **when** $\nu\to 0$ **(or,** which is the same thing, when $\beta\to 0$), then with $\rho<n$ will be obtained formulas (6.10)of the previous paragraph, i.e., "impatient" claims will become "patient."

8. The closed systems of mass maintenance.

Until now, we examine such systems of mass maintenance, where the claims come from somewhere from without and the intensity of flow of claims does not depend on the state of system itself. In present

paragraph we will consider mass other other mass - such systems, in which the intensity of flow of the entering claims depends on the state of the very SMO. Such systems of mass maintenance are called locked.

As an example locked of SMO, let us consider following system. A working-repairman service/maintains n of machine tools. Each machine tool can at any moment leave the system and require servicing repairman's on the part. The intensity of flow of the malfunctions of each machine tool is equal to λ. The left the system machine tool stops. If at this moment worker is free, he is taken for tooling; on this he expends the mean time

$$\bar{t}_{o6} = \frac{1}{\mu},$$

where μ - an intensity of flow of maintenance (adjustments).

If at the moment of the failure of machine tool worker is occupied, machine tool stops in turn for maintenance and awaits until worker is freed.

It is required to find the probabilities of the states of this system and its characteristic:

- probability that the worker will not be occupied.

- the probability of the presence of turn.

- the average number of machine tools, which expect turn to repair, etc.

Before us - the peculiar system of the mass maintenance, where the sources of claims are the machine tools, available in the limited quantity and the feeding or not feeding claims depending on its state: on leaving of machine tool from system it ceases to be the source of new claims. Consequently, the intensity of the common/general/total flow of claims with which it is necessary to deal for worker, depends on that, how much is defective machine tools, i.e., how many claims are connected with the process of the maintenance (directly it is service/maintained or stands in turn).

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Characteristic for the closed system of mass maintenance is the presence of the limited number of sources of claims.

In essence, any SMO deals only with the limited number of sources of claims, but in a series of the cases, the number of these sources so is great, that may disregard the effect of the state of the very of SMO on the flow of claims. For example, the flow brought

about by ATS of large city proceeds, in essence, from subscribers's limited number, but this number so is great, that virtually it is possible to count the intensity of flow of the claims of independent of states ATS itself (how many channels are occupied at given torque/moment). In the locked system of mass maintenance the sources of claims, together with the channels of maintenance, are considered as cell/elements of SMO.

Let us consider the formulated above problem of a working-repairman within the framework of the common/general/total circuit of Markov processes.

The system, which includes worker and n of machine tools, has a series of the states which we will label according to the number of defective machine tools (machine tools, connected with maintenance):

S_0 - all machine tools are exact (worker is free).

S_1 - one machine tool is defective, worker is occupied with its adjustment.

S_2 - two machine tools are defective, one is put right, another expects turn.

. . . .

S_n - all n of machine tools are defective, one is put right, n - 1 stand in turn.

The graph/count of states is given to Fig. 5.9. The intensities of flow of the events, which translate system from state into state, are written of arrow/pointers. From state S_0 into S_1 the system translates the flow of malfunctions of all working machine tools; its intensity is equal to $n\lambda$. From state S_1 into S_2 system translates the flow of malfunctions no longer n, but n-1 by machine tool in (they work in all n-1) and so forth. As concerns the intensities of flow of events, which translate system on arrow/pointers from right to left, then they continually are identical - it works always they working with the intensity of maintenance μ.

Using, as usual, by the general solution of the problem of the maximum probabilities of states for the circuit of death and multiplication (§8 chapters 4), let us write the maximum probabilities of the states:

$$\left.\begin{array}{l} p_1=\frac{n\lambda}{\mu}p_0, \\ p_2=\frac{n(n-1)\lambda^2}{\mu^2}p_0, \\ \dots\dots\dots\dots\dots\dots \\ p_n=\frac{n(n-1)(n-2)\dots 1\,\lambda^n}{\mu^n}p_0, \\ p_0=\frac{1}{1+n(\lambda/\mu)+n(n-1)(\lambda/\mu)^2+\dots+n(n-1)\dots 1\cdot(\lambda/\mu)^n}. \end{array}\right\}$$

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Introducing, as before, designation $\lambda/\mu = \rho$, let us rewrite these formulas in the form:

$$\left.\begin{aligned}
p_0 &= \frac{1}{1+n\rho+n(n-1)\rho^2+\ldots+n(n-1)\ldots 1\cdot\rho^n},\\
p_1 &= n\rho p_0,\\
p_2 &= n(n-1)\rho^2 p_0,\\
&\ldots\ldots\ldots\ldots\ldots\\
p_n &= n(n-1)\ldots 1\rho^n p_0.
\end{aligned}\right\} \quad (8.1)$$

Thus, the probabilities of states SMO are found.

By virtue of the peculiarity of closed SMO, characteristic of its efficiency, they will be different from those which we applied earlier for SMO with the unlimited quantity of sources of claims.

The role of "absolute capacity" in this case will play the mean quantity of malfunctions, removed by worker per unit time. Let us compute this characteristic. Worker is occupied with tooling with the probability

$$P_{\text{зан}} = 1 - p_0. \quad (8.2)$$

If it is occupied, it service/maintains μ the machine tools (eliminates μ malfunctions) per unit time; that means the absolute capacity of the systems

$$A=(1-p_0)\mu. \qquad (8.3)$$

Relative capacity for locked SMO we do not compute, since each claim, after all, will be serviced: q = 1.

Probability that that worker will not be occupied:

$$P_{\text{своб}}=1-P_{\text{зан}}=p_0. \qquad (8.4)$$

Let us compute the average number of defective machine tools, otherwise - the average number of machine tools, connected with the process of maintenance. Let us designate this average number $\bar{w}$. Generally speaking, value $\bar{w}$ can be computed directly according to the formula

$$\bar{w}=1\cdot p_1+2\cdot p_2+\ldots+n\cdot p_n,$$

but it will simpler find it through absolute capacity A.

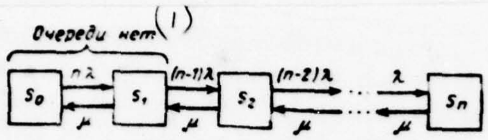

Fig. 5.9.

Key: (1). There is no turn.

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Actually, each working machine tool generates the flow of malfunctions with intensity λ; in our SMO on the average works $n - \bar{w}$ of machine tools; the generated or average/mean flow of malfunctions will have average/mean intensity $(n - \bar{w})\lambda$; all these malfunctions are removed by the worker, therefore,

$$(n-\bar{w})\,\lambda = (1-p_0)\,\mu,$$

whence

$$\bar{w} = n - \frac{\mu}{\lambda}\,(1-p_0)$$

or

$$\bar{w} = n - \frac{1-p_0}{\rho} \qquad (8.5)$$

Let us determine now the average number of machine tools $\bar{r}$, which expect adjustment in turn. Let us discuss as follows: the total number of machine tools W, connected with maintenance, is composed of the number of machine tools R, which stand in turn, plus the number of machine tools Ω, which are directly located under the maintenance:

$$W = R + \Omega.$$

The number of machine tools Ω, which are located under maintenance, is equal to one, if it is free, i.e., average value Ω is equal to the probability of the fact that the worker is occupied:

$$\bar{\omega}=1-p_0.$$

Subtracting this value from the average number $\bar{w}$ of the machine tools, connected with maintenance (defective), we will obtain the average number of machine tools, which expect maintenance in the turn:

$$\bar{r}=n-\frac{1-p_0}{\rho}-(1-p_0)=n-(1-p_0)\left(1+\frac{1}{\rho}\right). \quad (8.6)$$

Let us pause to one additional characteristic of the efficiency of SMO: at the productivity of the group of the machine tools, operated by worker.

Knowing the average number of defective machine tools $\bar{w}$ and productivity l of exact machine tool for time unit, it is possible to consider the average loss L of the productivity of the group of machine tools per unit time because of the malfunctions:

$$L=\bar{w}l.$$

Example 1. Worker service/maintains group of three machine tools. Each machine tool is stopped on the average of 2 times in hour. Alignment procedure occupies at worker, on the average, 10 minutes to determine the characteristics of locked SMO: the

probability of the employment of worker; his absolute capacity A; an average quantity of defective machine tools; the average relative loss of efficiency of the group of machine tools because of malfunctions.

Solution. We have $n=3,\ \lambda=2,\ \mu=\frac{1}{\bar{t}_{\text{об}}}=\frac{1}{1/6}=6,\ \rho=\lambda/\mu=1/3.$

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On formulas (8.1)

$$p_0=\frac{1}{1+3\cdot 1/3+3\cdot 2\cdot 1/3^2+3\cdot 2\cdot 1\cdot 1/3^3}\approx 0{,}346.$$

Probability of the employment of worker:

$$P_{\text{зан}}=1-p_0=0{,}654.$$

The absolute capacity of worker (average number of malfunctions which it eliminates in hour):

$$A=0{,}654\cdot 6=3{,}94.$$

The average number of defective machine tools we find through formula (8.5):

$$\bar{w}=3-\frac{0{,}654}{1/3}=1{,}04.$$

The average relative loss of efficiency of the group of machine tools because of malfunctions $\bar{w}/n$ = 0.347, i.e., the calculation of malfunctions the group of machine tools loses about 35o/o of productivity.

Let us consider a now more common/general/total example of locked SMO: brigade from m workers service/maintains n of the machine tools ($m < n$). Is enumerable the state of the system:

очереди нет (1)

S_0 — все станки работают, рабочие не заняты, (2)
S_1 — один станок остановился, один рабочий занят, (3)
S_2 — два станка остановились, два рабочих заняты, (4)
. . . .
S_m — m станков остановились, все рабочие заняты, (5)
S_{m+1} — $m+1$ станок остановился, m из них налаживаются, один ждет очереди, (6)
. . . .
S_n — все n станков остановились, m из них налаживаются, $n-m$ ждут очереди. (7)

Key: (1). No turn. (2). all machine tools work, workers are not occupied. (3). one machine tool stopped, one worker was occupied. (4). two machine tools stopped, two workers were occupied. (5). machine tools they stopped, all workers were occupied. (6). machine tool stopped, m of them are put right, one awaits turn. (7). all n of machine tools stopped, m of them are put right n - m await turn.

The graph/count of the states of system is shown on Fig. 5.10 (intensities of flow of events are written of arrow/pointers).

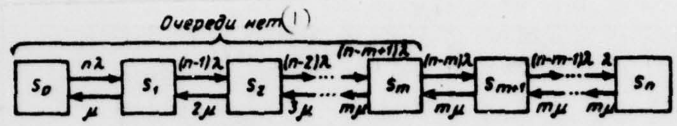

Fig. 5.10.

Key: (1). There is no turn.

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Applying common/general/total of the solution for the circuit of death and multiplication, we find the maximum probabilities of the states:

$$p_1 = \frac{n}{1}\,\frac{\lambda}{\mu}\,p_0,$$

$$p_2 = \frac{n(n-1)}{1\cdot 2}\left(\frac{\lambda}{\mu}\right)^2 p_0,$$

$$p_3 = \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\left(\frac{\lambda}{\mu}\right)^3 p_0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$p_m = \frac{n(n-1)(n-2)\ldots(n-m+1)}{1\cdot 2\cdot 3\ldots m}\left(\frac{\lambda}{\mu}\right)^m p_0,$$

$$p_{m+1} = \frac{n(n-1)\ldots(n-m)}{1\cdot 2\ldots m\cdot m}\left(\frac{\lambda}{\mu}\right)^{m+1} p_0,$$

$$p_{m+2} = \frac{n(n-1)\ldots(n-m-1)}{1\cdot 2\ldots m\cdot m^2}\left(\frac{\lambda}{\mu}\right)^{m+2} p_0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$p_n = \frac{n(n-1)\ldots 1}{1\cdot 2\ldots m\cdot m^{n-m}}\left(\frac{\lambda}{\mu}\right)^n p_0,$$

$$p_0 = \left[1 + \frac{n}{1}\,\frac{\lambda}{\mu} + \frac{n(n-1)}{1\cdot 2}\left(\frac{\lambda}{\mu}\right)^2 + \ldots + \frac{n(n-1)\ldots(n-m+1)}{1\cdot 2\ldots m}\left(\frac{\lambda}{\mu}\right)^m + \frac{n(n-1)\ldots(n-m)}{1\cdot 2\ldots m\cdot m}\left(\frac{\lambda}{\mu}\right)^{m+1} + \ldots + \frac{n(n-1)\ldots 1}{1\cdot 2\ldots m\cdot m^{n-m}}\left(\frac{\lambda}{\mu}\right)^n\right]^{-1}.$$

Designating, as ever, $\lambda/\mu=\rho$ let us lead formulas to the form:

$$
\left.
\begin{aligned}
p_0 &= \Big[1 + \frac{n}{1!}\rho + \frac{n(n-1)}{2!}\rho^2 + \ldots + \\
&\quad + \frac{n(n-1)\ldots(n-m+1)}{m!}\rho^m + \\
&\quad + \frac{n(n-1)\ldots(n-m)}{m!\,m}\rho^{m+1} + \ldots + \frac{n(n-1)\ldots 1}{m!\,m^{n-m}}\rho^n\Big]^{-1}, \\
p_1 &= \frac{n}{1!}\rho p_0, \\
p_2 &= \frac{n(n-1)}{2!}\rho^2 p_0, \\
p_3 &= \frac{n(n-1)(n-2)}{3!}\rho^3 p_0, \\
&\ldots\ldots\ldots\ldots\ldots\ldots \\
p_m &= \frac{n(n-1)\ldots(n-m+1)}{m!}\rho^m p_0, \\
p_{m+} &= \frac{n(n-1)\ldots(n-m)}{m!\,m}\rho^{m+1} p_0, \\
p_{m+2} &= \frac{n(n-1)\ldots(n-m-1)}{m!\,m^2}\rho^{m+2} p_0, \\
&\ldots\ldots\ldots\ldots\ldots\ldots \\
p_n &= \frac{n(n-1)\ldots 1}{m!\,m^{n-m}}\rho^n p_0.
\end{aligned}
\right\} \quad (8.7)
$$

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Through these probabilities is expressed the average number $\bar{z}$ of the occupied workers:

$$
\begin{aligned}
\bar{z} &= 0\cdot p_0 + 1\cdot p_1 + 2\cdot p_2 + \ldots + m\cdot(p_m + p_{m+1} + \ldots + p_n) = \\
&p_1 + 2p_2 + \ldots + (m-1)p_{m-1} + m(1 - p_0 - p_1 - \ldots - p_{m-1}).
\end{aligned} \quad (8.8)
$$

Through $\bar{z}$ is expressed, in turn, the average number of machine tools, operated by the team per unit time (absolute capacity):

$$A = \bar{z}\mu, \quad (8.9)$$

and also the average number of defective machine tools:

$$\bar{w}=n-\frac{\bar{z}\cdot\mu}{\lambda}=n-\frac{\bar{z}}{\rho}. \tag{8.10}$$

Hence is located the average loss of efficiency of the group of machine tools per unit time due to malfunctions: it is necessary to multiply the number of defective machine tools $\bar{w}$ by the productivity of one machine tool per unit of time.

Example 2. Two Workers they service/maintain group of six machine tools. The cessations of each (worker) machine tool occur, on the average, through each of half-hour. Alignment procedure occupies at worker on the average of 10 minutes to determine the characteristics of locked SMO:

- average number of occupied workers,

- an absolute capacity,

- an average quantity of defective machine tools.

Solution. We have: $n=6,\ m=2,\ \lambda=2,\ \mu=1/\bar{t}_{o6}=6,\ \rho=\lambda/\mu=1/3.$ On formulas (8.7)

$$p_0=\left(1+\frac{6}{1}\cdot\frac{1}{3}+\frac{6\cdot5}{1\cdot2}\cdot\frac{1}{3^2}+\frac{6\cdot5\cdot4}{1\cdot2\cdot2}\frac{1}{3^3}+\frac{6\cdot5\cdot4\cdot3}{1\cdot2\cdot2^2}\cdot\frac{1}{3^4}+\right.$$

$$\left.+\frac{6\cdot5\cdot4\cdot3\cdot2}{1\cdot2\cdot2^3}\cdot\frac{1}{3^5}+\frac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{1\cdot2\cdot2^4}\cdot\frac{1}{3^6}\right)^{-1}=\frac{1}{6,549}\approx0,153,$$

$$p_1\approx6/1\cdot1/3\cdot0,153\approx0,306.$$

Hence average number of occupied workers:

$$\bar{z}=1p_1+2(1-p_0-p_1)=1\cdot0,153+2\cdot0,541\approx1,235.$$

Through formula (8.9) we find the absolute capacity

$$A=1,235\cdot 6=7,41.$$

Through formula (8.10) we find the average number of defective machine tools.

$$\bar{w}=6-7,41/2=2,295.$$

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9. Systems of mass maintenance with "mutual assistance" between channels.

Until now, we examined only similar SMO, in which each request can be service/maintained only by one channel; the empty channels cannot "help" that occupied in maintenance.

Generally, this not always is as follows: are encountered the systems of the mass maintenance where one and the same claim can be service/maintained simultaneously by two and more by channels. For example, one and the same left the system machine tool they can service/maintain two workers immediately. This "mutual assistance" between channels can occur both in open and in closed SMO.

In the examination of SMO with the mutual assistance between

channels, it is necessary to consider two factors:

1. How is speeded up the maintenance of the claim when above it does work not one, immediately several channels?

2. Which "discipline of mutual assistance", i.e., when and as several channels is taken on itself servicing of the very same request?

Let us first examine the first question. If we assume that on servicing the request works not one channel, but several (k) channels, intensity of flow of maintenance will not decrease with increase in k, i.e., it will represent by itself certain nondecreasing function of number k of working channels. Let us designate this function $\mu(k)$. The possible form of the function $\mu(k)$ is shown on Fig. 5.11.

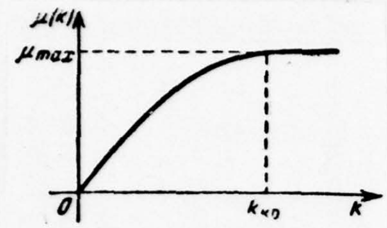
It is obvious that the unlimited increase in the number simultaneously of working channels not always does lead to a proportional increase in the speed of servicing; it is more natural to assume that at certain critical value $k = k_{\kappa p}$ further increase in the number of occupied channels no longer increases the intensity of maintenance.

In order to analyze work of SMO with the mutual assistance between channels, it is necessary, first of all, to assign the form of the function $\mu(k)$.

The simplest for research will be the case when function $\mu(k)$ grows proportionally k when $k < k_{кр}$, and when $k > k_{кр}$ it remains constant and equal to $\mu_{max} = k_{кр}\mu$ (see Fig. 5.12). If in this case the total number of channels n, which can help each other, does not exceed $k_{кр}$:

$$n \leqslant k_{кр},$$

then it is possible to consider the intensity of the maintenance of claim several channels of the proportional to the number of channels.



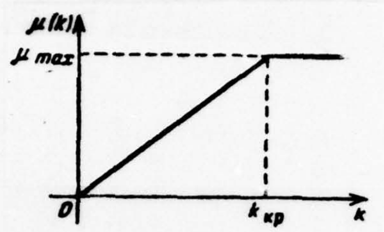


Fig. 5.12.

Fig. 5.11.

Fig. 5.12.

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Let us pause now at the second question: a discipline of mutual assistance. The simplest case of this discipline we will designate conditionally "whole as one". This means that during the appearance of its one claim they begin to service/maintain all n of channels immediately and remain occupied until is finished the maintenance of this claim; then all channels are changed over to the maintenance of another claim (if it there is) or they await its appearance, if it does not exist, and so forth. It is obvious, in this case all n of channels work as one, SMO becomes single-channel, but with the higher intensity of maintenance.

Does arise the question: to profitably or disadvantageously

introduce this mutual assistance between channels? Answer/response to this question depends on which the intensity of flow of claims, which form of the function $\mu(k)$, which type of SMO (with failures, with turn), which value is chosen as the characteristic of the efficiency of maintenance.

Example 1. There is three-channel SMO with failures: the intensity of the flow of claims $\lambda = 4$ (claim per minute), the mean servicing time of one claim by one channel $\bar{t}_{об}=0,5$ (min), function $\mu(k) = k\mu$. It does ask itself, is profitable from the point of the capacity of SMO to introduce the mutual assistance between channels according to type "everything as one"? Is profitable whether this from the point of the decrease of average time of the stay of claim in system?

Solution. a) without the mutual assistance

$$n = 3,\ \lambda = 4,\ \mu = 1/0,5 = 2,\ \rho = \lambda/\mu = 2.$$

According to the Erlang formulas (see § 4) we have:

$$p_0 = \frac{1}{1+\frac{2}{1!}+\frac{2^2}{2!}+\frac{2^3}{3!}} = \frac{3}{19} \approx 0,158;$$

$$P_{отк} = p_3 = \frac{2^3}{3!}\,p_0 = \frac{4}{3}\,\frac{3}{19} \approx 0,21.$$

Relative spacing capacity of SMO:

$$q = 1 - P_{отк} \approx 0,79.$$

Absolute capacity:

$$A = \lambda q \approx 4 \cdot 0,79 = 3,16.$$

The mean retention time of claim in SMO will be found as probability that the claim will be accepted for maintenance, multiplied by mean servicing time:

$$\bar{t}_{\text{сист}}=0{,}79\cdot 0{,}5=0{,}395 \text{ (мин)}.(1)$$

Key: (1). min.

It is not necessary to forget, what this mean time is related by KO to all claims - both serviced and not serviced us it can interest the mean time which will stay in system the serviced claim. This time is equal to:

$$\bar{t}^{(\text{об})}_{\text{сист}}=\bar{t}_{\text{об}}=0{,}5 \text{ (мин)}.(1)$$

Key: (1). min.

b) with the mutual assistance

$$n^{*}=1,\quad \lambda=4,\quad \mu^{*}=3\mu=6,\quad \rho^{*}=\frac{\lambda}{\mu^{*}}=\frac{2}{3};$$

$$p_0=\frac{1}{1+2/3}=\frac{3}{5};\quad p_1=\frac{2}{3}\cdot\frac{3}{5}=\frac{2}{5};$$

$$P_{\text{отк}}=p_1=\frac{2}{5};\quad q=1-\frac{2}{5}=\frac{3}{5}=0{,}6$$

$$A=\lambda q=4\cdot 0{,}6=2{,}4.$$

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Mean retention time of claim in SMO:

$$\bar{t}_{\text{сист}}=p_1\cdot 1/3\mu=2/5\cdot 6=0{,}0667 \text{ (min)}.$$

Mean retention time of the serviced claim in SMO:

$$\bar{t}^{(об)}_{сист} = 1/3\mu = 0,167 \text{ (мин)}(1)$$

Key: (1). min.

Thus, in the presence of mutual assistance "whole as one" capacity of SMO noticeably decreased. This is explained by an increase in the failure probability: for the time, when all channels are occupied with the maintenance of one claim, can arrive other claims, and, it is logical, to obtain failure. As concerns the mean retention time of claim in SMO, then it, as one would expect that it decreased. If, for some reasons, we strive by KO to the all possible decrease of time which the claim carries out in SMO (for example, if the stay in SMO it is dangerous for a claim), it can seem that, in spite of the decrease of capacity, all the same it will profitably join three channels into one.

Let us consider now the effect of a mutual assistance of the type "whole as one" on work of SMO with expectation. Let us take for simplicity only case of the unlimited turn. It is logical, the effects of mutual assistance on capacity of SMO in this case will not be, since under any conditions serviced will be all the come claims. Arises the question concerning the effect of mutual assistance on the

characteristics of the expectation: the average length of turn, mean latency, mean retention time in SMO.

By virtue of of formulas (6.13), (6.14) § 6 for maintenance without mutual assistance the average number of claims in turn will be

$$\bar{r}=\frac{\rho^{n+1}p_0}{n\cdot n!\,(1-\varkappa)^2}, \qquad (9.1)$$

average time of the expectation:

$$\bar{t}_{\text{ож}}=\frac{\rho^n p_0}{n\mu\cdot n!\,(1-\varkappa)^2}, \qquad (9.2)$$

a mean retention time in the system:

$$\bar{t}_{\text{смст}}=\bar{t}_{\text{ож}}+1/\mu, \qquad (9.3)$$

where

$$p_0=\frac{1}{1+\frac{\rho}{1!}+\frac{\rho^2}{2!}+\dots+\frac{\rho^n}{n!}+\frac{\rho^{n+1}}{n!\,(n-\rho)}}, \qquad \varkappa=\frac{\rho}{n}. \qquad (9.4)$$

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But if is applied a mutual assistance of the type "whole as one", then system will work as single-channel with the parameters

$$\rho^*=\lambda/\mu^*=\lambda/n\mu=\rho/n=\varkappa$$

and its characteristics will be determined by formulas (5.14), (5.15) §5:

$$\bar{r} = \frac{\varkappa^2}{1-\varkappa}. \quad (9.5)$$

$$\bar{t}_{\text{ож}} = \frac{1}{n\mu} \frac{\varkappa}{1-\varkappa}. \quad (9.6)$$

$$\bar{t}_{\text{сист}} = \bar{t}_{\text{ож}} + \frac{1}{n\mu} = \frac{1}{n\mu(1-\varkappa)}. \quad (9.7)$$

Example 2. There is three-channel SMO with unlimited turn; intensity of the flow of claims λ = 4 (claims in min.) mean servicing time $\bar{t}_{\text{об}} = 0{,}5$ (min). Function $\mu(k) = k\mu$ ($k_{\text{кр}} > 3$). It is profitable whether, keeping in mind:

- average length of turn,

- mean latency of maintenance,

- mean retention time of claim in SMO

does introduce the mutual assistance between channels of the type "everything as one"?

Solution. a) without mutual assistance.

$$n = 3,\ \lambda = 4,\ \mu = 1/0{,}5 = 2,\ \rho = \lambda/\mu = 2,\ \varkappa = \rho/n = 2/3.$$

On formulas (9.1)-(9.4) we have

$$p_0 = \frac{1}{1+\frac{2}{1!}+\frac{2^2}{2!}+\frac{2^3}{3!}+\frac{2^4}{3!\,(3-2)}} = \frac{1}{9};$$

$$\bar{r} = \frac{2^4\cdot 1/9}{3\cdot 3!\,(1/3)^2} = \frac{8}{9} = 0,889;$$

$$\bar{t}_{\text{ож}} = \bar{r}/\lambda = 2/9 = 0,222;$$

$$\bar{t}_{\text{сист}} = \bar{t}_{\text{ож}} + \bar{t}_{\text{об}} = 2/9 + 1/2 = 0,722.$$

b) with the mutual assistance

$$n^* = 1,\quad \lambda = 4,\quad \mu^* = 3\mu = 6,\quad \rho^* = \lambda/\mu^* = \varkappa = 2/3.$$

Through formulas (9.5)-(9.7) we find:

$$\bar{r} = \frac{(2/3)^2}{1/3} = \frac{4}{3} = 1,333;$$

$$\bar{t}_{\text{ож}} = \frac{1}{6}\,\frac{2/3}{1/3} = \frac{1}{3} = 0,333;$$

$$\bar{t}_{\text{сист}} = \bar{t}_{\text{ож}} + \bar{t}_{\text{об}} = 1/3 + 1/6 = 0,500.$$

Thus, the average length of turn and mean latency in turn in the case of mutual assistance, are more, but the mean retention time of claim in system - is less.

From the examined examples it is evident that the mutual assistance between channels of the type "whole as one", as a rule, does not contribute to the increase of the efficiency of the maintenance: the retention time of claim in SMO is reduced, but deteriorate other characteristics of maintenance.

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Therefore it is desirable to change the discipline of servicing so that the mutual assistance between channels would not interfere to accept to maintenance new claims, if they appear for time, when all channels are occupied.

Let us name conditionally "uniform mutual assistance" the following type of mutual assistance. If claim comes at the torque/moment when all channels are free, then whole n of channels they are accepted for its servicing; if, at the moment of maintain/serviceing the claim, comes one additional, the part of the channels is changed over to its maintenance; if for the moment are service/maintained these two claims, comes one additional, the part of the channels is changed over to its maintenance and so forth, until render/show occupied all n of channels; if this then, the newly come claim obtains failure (in SMO with failures) or it stops in turn (in SMO with expectation).

With this discipline of mutual assistance, the claim obtains failure or it stops in turn only if there is no its possibility to service. As concerns "idle time" of channels, then it under these conditions is minimal: if in system is at least one claim, all channels work.

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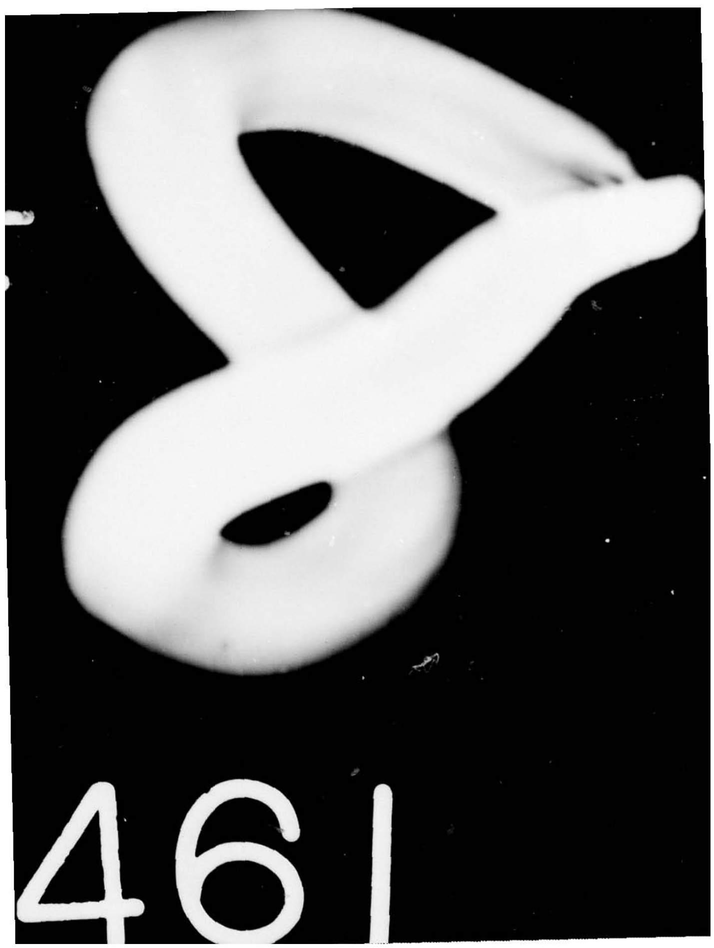
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Above we mentioned that during the appearance of a new claim the part of the occupied channels is free/released and is changed over to the maintenance of the newly arrived claim. What part? This depends on the form of the function $\mu(k)$. If it takes the form of linear dependence, as shown in Fig. 5.12, and $k_{\text{кр}} > n$, then it does not matter what part of the channels to isolate into the maintenance of the newly acted claim, provided all channels were occupied (then the total intensity of maintenance during any distribution of channels according to claims will be equal to $n\mu$). It is possible to demonstrate that if the curve $\mu(k)$ is convex upwards, as shown in Fig. 5.11, then it is necessary to distribute channels on claims as possible more evenly.

Let us consider the work of n-channel SMO with the "uniform" mutual assistance between channels.

1. SMO with failures.

Let us label the states of SMO according to the number of claims, which are found in the state of servicing:

S_0 - SMO is free,

S_1 - one claim is service/maintained by all n by channels.

S_2 - two claims are service/maintained by all n by channels,

. . . .

$S_k - k$ - claims they are service/maintained by all n by channels.

. . . .

$S_n - n$ claims they are service/maintained by all n by channels.

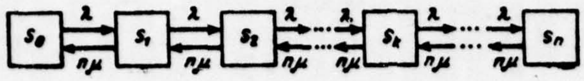

Fig. 5.13.

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We see that the graph/count of the states (see Fig. 5.13) here the same as for single-channel SMO with productivity $\mu^* = n\mu$ and the limited turn, which has n - 1 places. Therefore for determining of the system characteristics, we can use formulas § 5, substituting in them $\varkappa = \lambda/\mu^* = \lambda/n\mu$ for $\rho = \lambda/\mu$:

$$P_{\text{отк}} = \frac{\varkappa^n (1-\varkappa)}{1-\varkappa^{n+1}}; \qquad (9.8)$$

$$q = \frac{1-\varkappa^n}{1-\varkappa^{n+1}}; \qquad (9.9)$$

$$A = \lambda q = \lambda \frac{1-\varkappa^n}{1-\varkappa^{n+1}}. \qquad (9.10)$$

Example 3. Under conditions of example 1 to compare the relative and absolute capacity of SMO, and also the average number of occupied channels:

a) in the absence of mutual assistance,

~~b)~~ c) in the presence of the uniform mutual assistance between channels.

Solution a) without mutual assistance.

From example 1, we have q = 0.79, A = 3.16. Average number of occupied channels $\bar{z} = A/\mu = 1.58$.

b) with uniform mutual assistance.

$$\varkappa = \lambda/\mu n = 4/2\cdot 3 = 2/3.$$

On formula (9.9)

$$q = \frac{1-(2/3)^3}{1-(2/3)^4} \approx 0,887; \quad A = 4q \approx 3,51; \quad \bar{z} = 3,51/2 \approx 1,76.$$

Thus because of the application/use of the reasonably organized mutual assistance between channels, the capacity of SMO somewhat increased. With respect was increased the average employment of channels.

2. SMO with turn.

Let us consider SMO with turn and the maximum number of claims in turn m. Let us assume that between channels is a "uniform" mutual assistance and $\mu(k) = k\mu$. The states of system again let us label according to the number of claims, located in SMO:

S_0 - system is free,

S_1 - one claim is service/maintained by all n by channels,

S_2 - two claims are service/maintained by all n by channels,

. . . .

$S_k - k$ claims they are service/maintained by all n by channels, turn there are no,

. . . .

$S_n - n$ claims they are service/maintained by all n by channels, turn there are no,

$S_{n+1} - n$ claims are service/maintained by all n by channels, one claim stands in turn,

. . . .

$S_{n+m} - n$ claims they are service/maintained by all n by channels, in turn stands m of claims.

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The graph/count of states of SMO is given to Fig. 5.14.

We again obtained the graph/count of the same form, as in Fig. 5.13, but with the increased by m number of states. That means that

we should use formulas § 5 for single-channel SMO with productivity $\mu^* = n\mu$ and the number of places in turn n + m - 1. We will obtain:

$$P_{\text{отк}} = \frac{\varkappa^{n+m}(1-\varkappa)}{1-\varkappa^{n+m+1}}; \tag{9.11}$$

$$q = \frac{1-\varkappa^{n+m}}{1-\varkappa^{n+m+1}}; \tag{9.12}$$

$$A = \lambda q = \lambda \frac{1-\varkappa^{n+m}}{1-\varkappa^{n+m+1}}. \tag{9.13}$$

Example 4. Under conditions of example 1 to compare absolute and relative capacities for the case of the absence of mutual assistance and presence of uniform mutual assistance, if in turn is two places (m =2).

Solution a) without mutual assistance. From example 1, we have q ≈ 0.79; A ≈3.16.

b) with uniform mutual assistance.

On formulas (9.11)-(9.13) for $n=3,\ \lambda=4,\ \mu=2,\ \rho=2,\ \varkappa=\rho/n=2/3$ we have:

$$q = \frac{1-(2/3)^5}{1-(2/3)^6} = \frac{57}{65} = 0{,}88, \quad A = \lambda q = 3{,}52.$$

We let the reader independently count the average number of claims in turn, mean latency and mean retention time in turn, mean latency and mean retention time in the system for both versions of example of 4 and to ascertain that in the presence of the uniform

mutual assistance between channels all the characteristics of SMO vary only in desirable for us direction.

One ought not, however, to forget that organization of this mutual assistance between channels by no means for all SMO it is realizable.

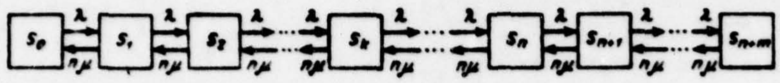

Fig. 5.14.

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10. System of mass servicing with errors.

Sometimes in practice it feels it is necessary to meet such cases when the claim, accepted for maintenance in SMO, is service/maintained not with full/total/complete authenticity, but with certain probability $p \neq 1$; in other words, they can occur of the errors in maintenance, result of which is the fact that some claims, passed SMO and allegedly "serviced", remain in actuality not serviced due to "reject" in the work of SMO.

Examples of SMO with errors can be: by reference bureaus, sometimes salient incorrect references and indications; corrector, capable of passing error or it is erroneous it to correct; exchange, which sometimes combines subscriber not with that number; the air defense system [PVO] for which "maintenance" is the bombardment of target/purpose, as is known, not always ending with its damage/defeat, etc.

In that extended SMO with errors the appearance of an error in maintenance does not virtually affect the flow of the claims: the number of sources of claims so greatly that intensity of flow as a result of error does not virtually vary. Therefore for the extended systems of mass maintenance, the account of the errors in maintenance is reduced only to the fact that the relative capacity of system is reduced: it is multiplied by $p < 1$, where p - probability of error-free maintenance. Respectively, is multiplied by p absolute capacity. As concerns remaining characteristics of SMO, such, for example, as latency, the number of claims in turn and so forth, on them the errors in maintenance do not manifest themselves. Another matter - for the closed system of the mass maintenance when the claim, serviced with error, again becomes in turn for maintenance, and therefore increases loading of SMO.

As an example of closed SMO with errors, let us consider one working, operating n of machine tools. The intensity of flow of the malfunctions of one working machine tool is equal to λ, the mean servicing time (adjustment) of machine tool $\bar{t}_{об} = 1/\mu$; with probability p of servicing is finished successfully, and machine tool begins again to work; with probability 1 - p maintenance proves to be unsuccessful, and machine tool again becomes in turn for maintenance.

It is required to determine the maximum probabilities of states.

Let us label the states of SMO according to the number of defective machines:

S_0 - all machine tools are exact,

S_1 - one machine tool is defective, is put right, awaits turn,

S_2 - two machine tools are defective, one is put right, another awaits in turn,

. . . .

S_k — k machine tools are defective, one is put right, k - 1 await turn,

S_n machine tools are defective one it is put right, n - 1 await turn.

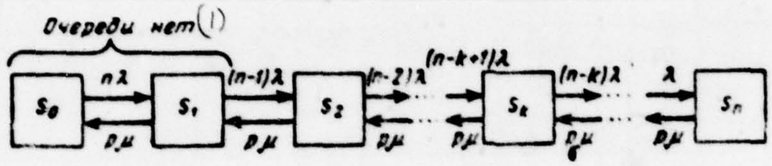

Fig. 5.15.

Key: (1). There is no turn.

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The graph/count of the states of system is shown on Fig. 5.15. The presence of the errors in maintenance manifests itself in the fact that of arrow/pointers, that goes from right to left, stands not the intensity of servicing μ, but the intensity of "successful maintenance" $p\mu$, where p - probability that servicing will be carried out successfully. Actually, let, for example, system be located in state S_k (one machine tool it is put right k - 1 they await turn). Probability that for time Δt will be finished servicing, is equal to $\mu\Delta t$; but this maintenance only with probability p will be successful and transfer the system from state S_k in S_{k-1}; with probability 1 - p it will be unsuccessful and claim again will return in turn, therefore, system again will remain in state S_k. That means that the intensity of flow of successful maintenance will be equal to $p\mu$, that

also is noted in Fig. 5.15. The obtained graph/count in no way differs from that which is given to Fig. 5.9 with that difference, which instead of μ on it stands $\mu^* = p\mu$. That means that the characteristics of SMO with errors can be calculated according to formulas § 8, with replacement μ by $p\mu$.

Example 1. Worker service/maintains group of three machine tools. The cessations of working machine tool occur on the average two times in hour. Alignment procedure takes away from worker on the average of 10 minutes, moreover malfunction is removed with probability by 2/3 (and remains unremoved with probability 1/3). To determine the characteristics of this closed SMO; the probability of the employment of worker, absolute capacity; an average quantity of defective machine tools.

Solution. For $n=3,\ \lambda=2,\ \mu=1/\bar{t}_{об}=6,\ p=2/3,\ \mu^*=p\mu=4,\ \rho^*=\lambda/\mu^*=1/2$ through formulas (8.1) we find

$$p_0=\frac{1}{1+3\cdot 1/2+3\cdot 2\cdot 1/2^2+3\cdot 2\cdot 1\cdot 1/2^3}\approx 0{,}211.$$

Probability of the employment of worker:

$$P_{зан}=1-p_0\approx 0{,}789.$$

Absolute capacity (number of malfunctions, removed worker in hour):

$$A=0{,}789\cdot 4\approx 3{,}16.$$

The average number of defective machine tools we find through formula

(8.5):

$$\bar{w} = 3 - \frac{0,789}{1/2} = 1,42.$$

The original case of SMO with errors represents such system of mass maintenance, in which the character of servicing depends on the length of the turn: with an increase by this is reduced the servicing time, but increases the probability of error. It goes without saying that this situation is created only where the "channel of servicing" is living person.

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Let us consider the example of similar SMO. Let us take locked single-channel SMO with n scurces of claims (working, servicing n machine tools). Let in the absence of turn (under normal conditions) the mean time of servicing be equal to $\bar{t}_{\text{об}}$, and it means the intensity of flow of maintenance it is equal to $\mu(0) = 1/\bar{t}_{\text{об}}$. When, in the turn, expecting are present, tooling, the worker begins to hurry, and the intensity of flow of servicings increases. Let us designate the intensity of flow of maintenance when, in turn, r the machine tools are present, through $\mu(r)$. Simultaneously with an increase in the rate of servicing (in connection with an increase of the number of machine tools, which expect in turn) increases the probability of error; in the absence of turn (under normal conditions) it is equal p(0), and when, in turn, r machine tools are present, - p(r).

Obviously it is necessary to multiply for each r the intensity of servicing and the probability of of error and introducing one "given" intensity of servicing:

$$\mu^*(r) = \mu(r) \cdot p(r) \quad (r = 0, \ldots, n-1).$$

The graph/count of states of SMO is represented in Fig. 5.16 (numbering of states - the same as it is above). Applying common/general/total formulas for maximum probabilities in the circuit of destruction and multiplication, we will obtain:

$$\left.\begin{aligned}
p_1 &= \frac{n\lambda}{\mu^*(0)} p_0, \\
p_2 &= \frac{n(n-1)\lambda^2}{\mu^*(0)\mu^*(1)} p_0, \\
p_3 &= \frac{n(n-1)(n-2)\lambda^3}{\mu^*(0)\mu^*(1)\mu^*(2)} p_0, \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
p_k &= \frac{n(n-1)\ldots(n-k+1)\lambda^k}{\mu^*(0)\mu^*(1)\ldots\mu^*(k-1)} p_0, \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
p_n &= \frac{n(n-1)\ldots 1 \cdot \lambda^n}{\mu^*(0)\mu^*(1)\ldots\mu^*(n-1)} p_0, \\
p_0 &= \left[1 + \frac{n\lambda}{\mu^*(0)} + \frac{n(n-1)\lambda^2}{\mu^*(0)\mu^*(1)} + \ldots + \right. \\
&\quad + \frac{n(n-1)\ldots(n-k+1)\lambda^k}{\mu^*(0)\mu^*(1)\ldots\mu^*(k-1)} + \ldots + \\
&\quad \left. + \frac{n(n-1)\ldots 1\cdot\lambda^n}{\mu^*(0)\mu^*(1)\ldots\mu^*(n-1)}\right]^{-1}
\end{aligned}\right| \qquad (10.1)$$

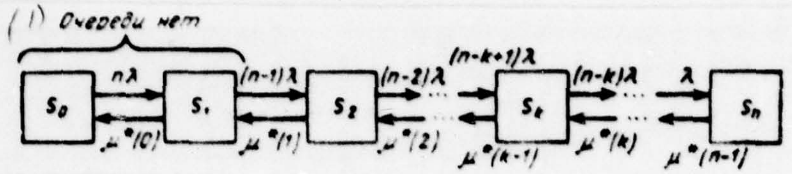

Fig. 5.16.

Key: (1). There is no turn.

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11. Systems of mass servicing with non-Poisson flows of events.

All the examined, until now, problems of queueing theory were related only to the case, when the process, taking place in SMO, represents continuous Markov circuit (Markov process with discrete states and the continuous time), in other words, when all flows of events, which translate system from state into state (flows of claims, maintenance, attendance/departures, etc.) are Poisson. For obtaining the maximum system characteristics in steady-state steady state, it was required so that these flows would be not only Poisson, but also simplest (with constant intensities).

In practice it very frequently proves to be that the flows of

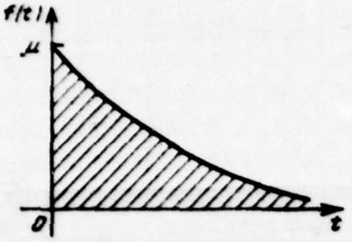
events, which function in the system of mass servicing, noticeably differ from protozoa. Especially this is related to the flow of servicings. It is real/actual, we know that in the simplest flow the time interval between two adjacent events is distributed according to the exponential law

$$f(t)=\mu e^{-\mu t} \quad (t>0).$$

(see Fig. 5.17). It is obvious that time T_{o6} of servicing of claim completely compulsorily is not distributed according to this law; on the contrary, much more typical is the case when the law of time allocation of servicing f(t) is different from the exponential, and its most probable value not is equal to zero (see Fig. 5.18).

In the case when the law of time allocation of maintenance is different from the exponential, increasingly previously examined methods of describing the processes, which take place in SMO, they stop, strictly speaking, unsuitable. In particular, it is not possible to register linear differential equations for the probabilities of states, by it linear algebraic equations for maximum probabilities. The mathematical apparatus of experiment becomes much more complex; analytical formulas for characteristics of SMO can be obtained only for the simplest cases.

Let us give (without proof) some of the obtained in this region results.



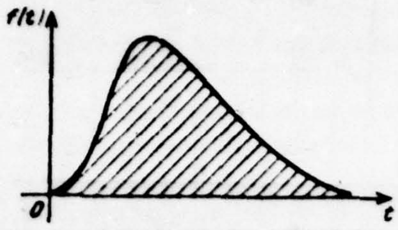


Fig 5.18.

Fig. 5.17

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1. SMO with failures.

Let to the n-channel system of mass servicing with failures enter the simplest flow of claims with intensity λ, and the time of servicing has arbitrary distribution with the mathematical expectation

$$\bar{t}_{o6} = \frac{1}{\mu} = \int_0^\infty t f(t)\, dt. \qquad (11.1)$$

Proved (see [16]), that in this case Erlang's formulas for the probabilities of states remain valid, namely

$$\left.\begin{aligned} p_k &= \frac{\rho^k}{k!} p_0, \quad (k = 0, \ldots, n), \\ p_0 &= \frac{1}{1 + \frac{\rho}{1!} + \frac{\rho^2}{2!} + \ldots + \frac{\rho^n}{n!}}. \end{aligned}\right\} \qquad (11.2)$$

where $\rho = \lambda/\mu = \lambda \bar{t}_{об}$.

2. Single-channel SMO with expectation.

Let there be the single-channel system of mass maintenance with the unlimited turn (n = 1, m = ∞); its entrance enters the simplest flow of claims with intensity λ; the law of time allocation of servicing f(t) - arbitrary, with mathematical expectation $\bar{t}_{об} = 1/\mu$ and root-mean-square deviation $\sigma_{t_{об}}$.

Value

$$\frac{\sigma_{t_{об}}}{\bar{t}_{об}} = v$$

is called the coefficient of a variation in the servicing time (this coefficient is shown, is how great time jitter of servicing relative to its mean value).

Proved (see for example, [20]), that for single-channel SMO with the simplest flow of claims and the arbitrarily distributed servicing time the average number of claims, which are located in turn, is

expressed by the formula:

$$\bar{r}=\frac{\rho^2(1+v^2)}{2(1-\rho)}, \qquad (11.3)$$

where $\rho = \lambda/\mu$, v -- a coefficient of a variation in the time of servicing. As concerns mean latency in turn, then it is expressed by the formula:

$$\bar{t}_{ож}=\frac{\rho^2(1+v^2)}{2\lambda(1-\rho)}.$$

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Formulas (11.3), (11.4) are usually called pol4ceka-Khinchin's formulas [20].

Let us note that for the exponential distribution

$$f(t)=\mu e^{-\mu t} \quad (t>0)$$

the coefficient of the variation

$$v=\frac{\sigma_{t_{об}}}{\bar{t}_{об}}=\frac{1/\mu}{1/\mu}=1.$$

In this case formulas (11.3) and (11.4) are converted into previously brought out by us formulas (5.17) and (5.20) (see § 5):

$$\bar{r}=\frac{\rho^2}{1-\rho}; \quad \bar{t}_{ож}=\frac{\rho^2}{\lambda(1-\rho)} \qquad (11.4)$$

Let us consider limiting case when the time of servicing not at all is not random and it is equal to its mathematical expectation:

$$\bar{t}_{об}=1/\mu.$$

Then $\sigma_{t_{об}}=0$, $v=0$, and formulas (11.3), (11.4) give

$$\bar{r}=\frac{\rho^2}{2(1-\rho)} \qquad (11.5)$$

and

$$\bar{t}_{\text{ож}} = \frac{\rho^2}{2\lambda(1-\rho)}, \qquad (11.6)$$

i.e. both average number of claims in turn and mean latency with the strictly time constant of servicing half than with the random servicing time, distributed according to exponential law.

Example 1. Flow of the trains, which enter the shunting station for treatment/working, the simplest flow with intensity $\lambda = 2$ (composition in hour). The mean time, spent on treatment/working of one composition, is equal to $\bar{t}_{\text{об}} = 20$ (min); its root-mean-square deviation $\sigma_{t_{\text{об}}} = 8$ (min). To determine the average number of compositions, which expect treatment/working and mean latency of treatment/working in turn, and also the average number of compositions, connected with maintenance at sorting station.

Solution. Passing to one and the same the unit of the measurement of time (hour) we have:

$$\mu = \frac{1}{\bar{t}_{\text{об}}} = \frac{1}{1/3} = 3 \text{ (составa в час)}^{(1)}$$

Key: (1). composition in hour.

Load factor (given intensity of flow claims station):

$$\rho = \lambda/\mu = 2/3.$$

Coefficient of a variation in the servicing time:

$$v = \sigma_{t_{\text{об}}}/\bar{t}_{\text{об}} = 8/20 = 0,4.$$

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Through formulas (11.3) and (11.4) we find the average number of compositions, which expect the treatment/working

$$\bar{r}=\frac{(2/3)^2\,(1+0,4^2)}{2\,(1-2/3)}\approx 0,77$$

and mean latency of the treatment/working:

$$\bar{t}_{\text{ож}}=\bar{r}/\lambda\approx 0,385\ (\text{час}).(1)$$

Key: (1). hour.

The average number of compositions, connected with shunting station, is equal to the average number of compositions in turn $\bar{r}$ plus the average number of compositions under servicing; the latter is equal to the probability of the employment of SMO, i.e., to the ratio of the average number of compositions, entering per unit time to the average number of compositions, operated by channel per unit time. The hence average number of compositions (claims) in system is equal to:

$$\bar{k}=\bar{r}+\rho=0,77+2/3\approx 1,437.$$

The given analytical formulas are related as was already said that to the simplest non-Poisson SMO. In the case of more complex SMO (multichannel, with the special feature/peculiarities of servicing, etc.), of simple analytical formulas it is impossible to obtain. In certain cases research of SMO with non-Poisson flows of events can be produced with the help of the method of pseudostates, described in §10 Chapter 4.

As an example let us consider single-channel SMO with turn (without limitations). The entrance of system, enters the simplest flow of claims with intensity λ; the time of servicing $T_{об}$ is distributed according to the law of Erlang of 2nd order with mathematical expectation $1/\mu$, i.e., it represents by itself the sum of two independent random quantities with identical exponential distribution. Let us designate the parameters of these exponential μ' distributions μ'. According to the theorem of the addition of mathematical expectations, we have:

$$1/\mu' + 1/\mu' = 2/\mu' = 1/\mu,$$

whence $\mu' = 2\mu$.

Thus, the servicing time $T_{об}$, distributed according to the law of Erlang of 2nd order with mathematical expectation $1/\mu$, can be represented as sum of two independent random values $T_{об}^{(1)}$ and $T_{об}^{(2)}$, which have each exponential distributicn with the parameter 2μ. These of two times $T_{об}^{(1)}$ and $T_{об}^{(2)}$ can be presented as two consecutive "phases" of the process of servicing.

Let us consider different states of SMO, labeling them according to the number claims in system and phase of servicing:

S_0 - there are no claims in system (maintenance does not occur);

$S_{1,1}$ - one claim is located in SMO, maintenance in the first phase, there is no turn;

$S_{1,2}$ - one claim is located in SMO, maintenance during the second phase, turn no;

$S_{2,1}$ - two claims are located in SMO, the first it is service/maintained (first phase), the second stands in turn;

$S_{2,2}$ - claim are located in SMO; the first is service/maintained (second phase), the second stands in turn;

. . . .

$S_{k,1}$ - k claims are located in SMO, one under servicing (first phase), the others - in turn;

$S_{k,2}$ - k claims are found in SMO, by one under maintenance (second phase) the others - in turn.

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The labeled graph/count of the states of system is given to Fig. 5.19. Real/actually from state S_0 into $S_{1,1}$ system translates the

flow of claims with intensity λ. From state $S_{1,1}$ into $S_{1,2}$ the system translates flow with intensity 2μ (flow of the terminations of the first phase of maintenance). From state $S_{1,2}$ in S_0 - the same flow. From state $S_{1,1}$ into $S_{2,1}$ the system translates the flow of claims, etc.

Using the labeled graph/count of states, let us register linear algebraic equations for the probabilities of the states:

$$\left.\begin{aligned}
&\lambda p_0 = 2\mu p_{1,2},\\
&(\lambda + 2\mu)\, p_{1,1} = \lambda p_0 + 2\mu p_{2,2},\\
&(\lambda + 2\mu)\, p_{1,2} = 2\mu p_{1,1},\\
&(\lambda + 2\mu)\, p_{2,1} = \lambda p_{1,1} + 2\mu p_{3,2},\\
&(\lambda + 2\mu)\, p_{2,2} = \lambda p_{1,2} + 2\mu p_{2,1},\\
&\dots\dots\dots\dots\\
&(\lambda + 2\mu)\, p_{k,1} = \lambda p_{k-1,1} + 2\mu p_{k+1,2},\\
&(\lambda + 2\mu)\, p_{k,2} = \lambda p_{k-1,2} + 2\mu p_{k,1},\\
&\dots\dots\dots\dots
\end{aligned}\right\} \qquad (11.7)$$

or introducing designation $\lambda/\mu = \rho$,

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$$\left.\begin{aligned}
&\rho p_0 = 2p_{1,2},\\
&(\rho + 2)\, p_{1,1} = \rho p_0 + 2p_{2,2},\\
&(\rho + 2)\, p_{1,2} = 2p_{1,1},\\
&(\rho + 2)\, p_{2,1} = \rho p_{1,1} + 2p_{3,2},\\
&\dots\dots\dots\dots\\
&(\rho + 2)\, p_{k,1} = \rho p_{k-1,1} + 2p_{k+1,2},\\
&(\rho + 2)\, p_{k,2} = \rho p_{k-1,2} + 2p_{k,1},\\
&\dots\dots\dots\dots
\end{aligned}\right\} \qquad (11.8)$$

This - the system of the infinite number of equations with the infinite number of unknowns p_0, $p_{1,1}$, $p_{1,2}$, $p_{2,1}$, $p_{2,2}$, ... There are methods, which make it possible to solve also systems literally, but they are comparatively complex, and we will not be on them stopped. We will be bounded to indication of that, as it can be solved (11.8) at the concrete/specific/actual values of the parameters λ and μ.

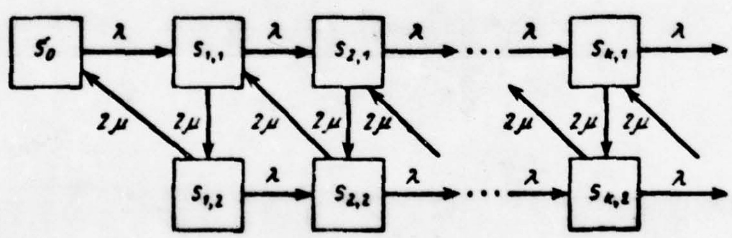

Fig 5.19.

First of all is considered the maximum virtually possible number of claims in turn - this can be done roughly, after taking the law of time allocation of maintenance exponential. When this is done, it is possible to reject/throw some (latter, beginning with some number) from equations, after which system (11.8) it is converted into the system of the finite number of equations with the finite number of unknowns, which is solved by the usual methods of the computational algebra (see, for example [21]). With the large number of equations, conveniently it is to use the method of iterations (successive approximations), moreover as the first approximation it is possible to take the values of the probabilities of states, obtained during exponential time allocation of maintenance after dividing probabilities equally between two phases of maintenance.

Applying the method of pseudo-states, it is possible, in principle, to approximately reduce any non-Markov process of mass maintenance to Markov; however with the large number of pseudostates the solution of the system of linear equations not only in literal, but also in numerical form it becomes hampered. In such cases for research of process, which takes place in SMO, it is possible to use the universal method of the simulation of random processes - by the so-called method of statistical testings (Monte Carlo) which will be examined in Chapter 8.

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6. METHOD OF DYNAMICS OF THE AVERAGES.

1. Idea of method, field of applicability.

In Chapters 4 and 5 we were introduced to the methods of describing the random processes, taking place in different physical systems, with the help of special mathematical apparatus - theory of continuous Markov chains. This apparatus makes it possible to comprise linear differential equations for the probabilities of states, and also linear algebraic equations for maximum probability of the states, which reflect the relative retention time of system in each of these states for the maximum, steady-state conditions/mode.

These methods represent by themselves convenient mathematical apparatus only in this case, when the number of possible states of system S is comparatively small. In the case when the number of possible states of system is great (order of several ten, and that also hundred), these methods cease to be convenient. First, the joint solution of large number not only differential, but also algebraic

equations it is difficult even in the presence of ETsVM [ЭЦВМ. digital computer]. Furthermore, even if to us it were possible to solve these equations and to find the probabilities of all states of systems, obtained results they will be difficultly foreseeable. In order to comprehend them, for us nevertheless it is necessary to use some generalized characteristics of process, some average values (by such, for example, as "average number of occupied channels" or the "average number of claims in the turns", which we used in queueing theory). Until now, we such average characteristics computed through the probabilities of states. However, in the case when states too much, this method becomes unacceptable.

Does arise the question: a it cannot be whether comprised and solved equations directly for us the average characteristics interesting, passing the probabilities of states? It turns out that it is possible - sometimes accurately, sometimes - approximately, with certain error. By such tasks is occupied the so-called "method of the dynamics of average". It places to itself with target/purpose the direct study of the average characteristics of the random processes, which take place in complex systems with the large (virtually boundless) number of states.

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It is interesting that the basic applicability of the method of the dynamics of average is precisely the fact that impedes the study of phenomena by the more detailed methods: the complexity of the studied processes and the large number of participating in them cell/elements. As everywhere, where are applied the methods of the probability theory, the mass character of the studied phenomena makes it possible to establish/install in them comparatively simple laws.

We will demonstrate the idea of the method of the dynamics of average based on following simplest example.

Let there be the complex physical system S, which consists of the large number N of uniform cell/elements (or the "units"), each of which can randomly pass from state into state. Let us assume that all flows of events, which translate system S (and each cell/element) from state into state - Poisson (although in the general case and not simplest, but with intensities, arbitrary form depending on time). Then the process, which takes place in system, is Markov.

Let us assume that each cell/element can be in any of n of the possible states:

$$\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_n,$$

and the state of system S at each torque/moment is characterized by

the number of cell/elements, which are found in each of the states. To us it is required to trace the random process, which takes place in system S.

In principle, it would be possible to use TU the methodology which we already applied earlier during the study of similar processes, namely, to consider all the possible states of system S:

$S_{N,0,\ldots,0}$ — all cell/elements are located in state $\mathcal{E}_1$, in other states there is not one cell/element;

$S_{N-1,1,0,\ldots,0}$ — one cell/element is located in state $\mathcal{E}_2$, all others - in state $\mathcal{E}_1$, and so forth and to find the probabilities of these states. However, with the large number of cell/elements N, even the enumeration of the possible states of system S is difficult, not that that composition and the solution of equations for the probabilities of states.

It is obvious, we should go by another way. We will be distracted from the possible states of system as a whole and will concentrate our attention in separate cell/element $\mathcal{E}$ (since all cell/elements are uniform, nevertheless, which this will be cell/element) and let us consider for it the graph/count of states (Fig. 6.1).

Let us introduce into examination random variable $X_k(t)$ — number of units, which are found at torque/moment t in state $\mathscr{E}_k$. Let us it call briefly the number of state $\mathscr{E}_k$ at torque/moment t. It is obvious, for any moment t, the sum of the numbers of all states is equal to the total number of cell/elements:

$$X_1(t)+X_2(t)+\ldots+X_n(t)=N,$$

cr, it is shorter:

$$\sum_{k=1}^{n} X_k(t)=N. \qquad (1.1)$$

The examined by us value $X_k(t)$ for any t represents by itself random variable, and generally, with varying t - the random function of time.

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Let us assign to itself the mission: to find for any t the fundamental characteristics of random variable $X_k(t)$ — its mathematical expectation

$$m_k(t)=M[X_k(t)] \qquad (1.2)$$

and the dispersion:

$$D_k(t) = D[X_k(t)]. \quad (1.3)$$

In other words, for each torque/moment of time t we wish to know the average value of the number of each state, and also the spread of actual number about average.

In order to find these characteristics, it is necessary to know the intensities of all flows of the events, which translate the cell/element (not system, namely element!) from state into state.

Let us assume that these intensities to us are known and written on the graph/count of the states (see Fig. 6.1). Then the number of each state $X_k(t)$ can be presented as sum of random variables each of which is connected with separate (i-m) cell/element, namely: is equal to one, if this cell/element at the moment of time t is located in state $\mathcal{E}_k$, and is equal to zero, if it is not located:

$$X_k^{(i)}(t) = \begin{cases} 1, \text{ если } i\text{-й элемент в момент } t \text{ находится в состоянии } \mathcal{E}_k; \ (1) \\ 0, \text{ если не находится. } (2) \end{cases} \quad (1.4)$$

Key: (1). if the i cell/element at torque/moment t is located in state $\mathcal{E}_k$. (2). if it is not located.

It is obvious, for any moment t, the total number of state S_k is equal to the sum of random variables (1.4):

$$X_k(t) = X_k^{(1)}(t) + X_k^{(2)}(t) + \ldots + X_k^{(N)}(t),$$

or it is shorter

$$X_k(t) = \sum_{i=1}^{N} X_k^{(i)}(t). \qquad (1.5)$$

If the intensities λ_{ij} of the flows of the events, which translate each cell/element from state into state, to us are known (and, ~~t.a.~~ that means not random), then values

$$X_k^{(1)}(t),\ X_k^{(2)}(t),\ \ldots,\ X_k^{(N)}(t)$$

for separate cell/elements were independent from each other.

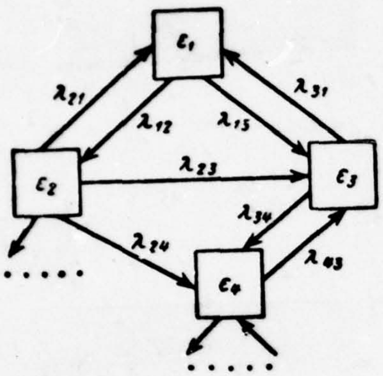

Fig. 6.1.

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According to the theorem of the addition of mathematical expectations (for which, by the way, independence it is not required) and the theorem of the addition of the dispersions:

$$\left.\begin{aligned} m_k(t) &= \sum_{i=1}^{N} M\left[X_k^{(i)}(t)\right], \\ D_k(t) &= \sum_{i=1}^{N} D\left[X_k^{(i)}(t)\right]. \end{aligned}\right\} \quad (1.6)$$

Let us find numerical characteristics - mathematical expectation and dispersion - random variable $X_k^{(i)}(t)$, by given one by expression (1.4). This value has two possible values: 0 and 1. The probability

of the first of them is equal to $p_k(t)$ — probability of the fact that the cell/element is located in state $\mathscr{E}_k$ (since cell/elements are uniform, then for all them this probability one and the same). A series of the distribution of each of random variables $X_k^{(i)}(t)$ one and the same takes the form:

$$\begin{array}{c|c} 0 & 1 \\ \hline 1-p_k(t) & p_k(t) \end{array}. \qquad (1.7)$$

where in upper row are shown the possible values of random variable, and in lower - their probability.

The mathematical expectation of random variable, assigned series of distribution (1.7), is equal to:

$$M\left[X_k^{(i)}(t)\right]=0\cdot\left(1-p_k(t)\right)+1\cdot p_k(t)=p_k(t),$$

where $p_k(t)$ — probability that the separate cell/element at torque/moment t will be located able $\mathscr{E}_k$. The dispersion of random variable with a series of distributions (1.7) is equal to:

$$D\left[X_k^{(i)}(t)\right]=\left(0-p_k(t)\right)^2\left(1-p_k(t)\right)+ \\ +\left(1-p_k(t)\right)^2 p_k(t)=p_k(t)\left(1-p_k(t)\right).$$

Substituting these expressions in formulas (1.6), let us find mathematical expectation and the dispersion of the number of k state:

$$m_k(t) = Np_k(t), \tag{1.8}$$
$$D_k(t) = Np_k(t)\left(1 - p_k(t)\right). \tag{1.9}$$

Thus, we succeeded in for any t finding mathematical expectation and dispersion of the number of any state $\mathscr{E}_k$: they are expressed by formulas (1.8) and (1.9) through the number of cell/elements N and the probability of the k state of any cell/element.

Knowing dispersion $D_k(t)$, it is possible to find the root-mean-square deviation of the number of state $\mathscr{E}_k$:

$$\sigma_k(t) = \sqrt{Np_k(t)\left(1 - p_k(t)\right)}, \tag{1.10}$$

and, which means, that for any moment of time t to indicate tentatively the range of the virtually possible values of the number:

$$m_k(t) \pm 3\sigma_k(t). \tag{1.11}$$

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Thus, without determining the probabilities of the states of system S as a whole, but being concerned only with probabilities of

the states of its separate cell/elements, it is possible to determine, to what is equal for any moment t the average number of each state and within which limits is located actual number. If we know the probabilities all states of one cell/element

$$p_1, p_2, \ldots, p_n$$

as of function of time, then to us are known the average numbers of states:

$$m_1, m_2, \ldots, m_n$$

and their dispersion:

$$D_1, D_2, \ldots, D_n$$

and the root-mean-square deviation:

$$\sigma_1, \sigma_2, \ldots, \sigma_n.$$

Thus, stated problem is reduced to the determination of the probabilities of the states of one separate cell/element.

These probabilities, as is known, can be found as solutions of the differential equations of Kolmogorov for the probabilities of states; the rules of their composition are given in §6 Chapter 4. For this, it is necessary to only know (accurately or approximately) the

intensities of flow of the events, which translate each cell/element from state into state. Thus far we will assume that these intensities to us are known and random. About from which considerations it is possible to determine these intensities, we will speak somewhat later (see §2).

Let us note that instead of the differential equations for the probabilities of states it is possible (it is sometimes more conveniently) to write equations it is direct for the average numbers of states. It is real/actual, as can be seen from formula (1.8), the average number of each state is proportional to the probability of this state (it differs from it in terms of factor N), and, obviously, satisfies the same differential equations, only integrated them must be under other initial conditions, which correspond initial by the numbers of states.

Example 1. System S consists of N of uniform cell/elements; the graph/count of the states of each cell/element is represented in Fig. 6.2. at the initial moment (with t = 0) all cell/elements are located in state s_1.

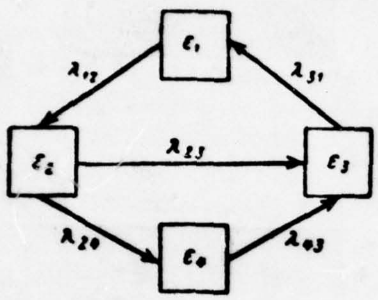

Fig. 6.2.

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To write the system of differential equations, by which they must satisfy the average numbers of states m_1, m_2, m_3, m_4, and to indicate that under its which initial conditions it is necessary to solve. Considering equations sclved, to write expressicns for the dispersions of the numbers of states.

Sclution. Directly on graph (Fig. 6.2) we compile an equation of Kolmogorov for the probabilities of the states:

$$
\left.\begin{aligned}
\frac{dp_1}{dt} &= -\lambda_{12} p_1 + \lambda_{32} p_3, \\
\frac{dp_2}{dt} &= -(\lambda_{23} + \lambda_{24}) p_2 + \lambda_{12} p_1, \\
\frac{dp_3}{dt} &= -\lambda_{31} p_3 + \lambda_{23} p_2 + \lambda_{43} p_4, \\
\frac{dp_4}{dt} &= -\lambda_{43} p_4 + \lambda_{24} p_2.
\end{aligned}\right\} \qquad (1.12)
$$

We know that one of these equations (any) it can be reject/thrown, but we thus far will preserve their everything.

Let us multiply the left and right side of each of equations (1.12) by the number of cell/elements N and will introduce in the left sides of N under the sign of derivative; we will obtain:

$$
\left.\begin{aligned}
\frac{d(Np_1)}{dt} &= -\lambda_{12} Np_1 + \lambda_{32} Np_3, \\
\frac{d(Np_2)}{dt} &= -(\lambda_{23} + \lambda_{24}) Np_2 + \lambda_{12} Np_1, \\
\frac{d(Np_3)}{dt} &= -\lambda_{31} Np_3 + \lambda_{23} Np_2 + \lambda_{43} Np_4, \\
\frac{d(Np_4)}{dt} &= -\lambda_{43} Np_4 + \lambda_{24} Np_2.
\end{aligned}\right\} \qquad (1.13)
$$

Let us now recall that

$$Np_1 = m_1, \quad Np_2 = m_2, \quad Np_3 = m_3, \quad Np_4 = m_4,$$

(argument t of these functions for brevity is reject/thrown) and let us rewrite equations (1.13) in the form:

$$\left.\begin{aligned} \frac{dm_1}{dt} &= -\lambda_{12} m_1 + \lambda_{31} m_3, \\ \frac{dm_2}{dt} &= -(\lambda_{23} + \lambda_{24}) m_2 + \lambda_{12} m_1, \\ \frac{dm_3}{dt} &= -\lambda_{31} m_3 + \lambda_{23} m_2 + \lambda_{43} m_4, \\ \frac{dm_4}{dt} &= -\lambda_{43} m_4 + \lambda_{24} m_2. \end{aligned}\right\} \quad (1.14)$$

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In equations (1.14) unknown functions are directly the average numbers of states. As is evident, these equations are comprised completely according to the same rule, as equation for the probabilities of states; therefore it was possible to comprise them immediately, passing intermediate stages (1.12) and (1.13). So we will enter subsequently.

It is obvious, for each t the average numbers of states satisfy the condition:

$$m_1 + m_2 + m_3 + m_4 = N,$$

and therefore one (any) of equations (1.14) it is possible to reject/throw. Let us reject/throw, for example, the third equation (it is most complicated) and into remaining equations instead of m_3 let us substitute the expression:

$$m_3 = N - (m_1 + m_2 + m_4).$$

Will be obtained finally the system of three differential equations:

$$\left.\begin{aligned}\frac{dm_1}{dt} &= -\lambda_{12} m_1 + \lambda_{31} [N - (m_1 + m_2 + m_4)],\\ \frac{dm_2}{dt} &= -(\lambda_{23} + \lambda_{24}) m_2 + \lambda_{12} m_1,\\ \frac{dm_4}{dt} &= -\lambda_{43} m_4 + \lambda_{24} m_2.\end{aligned}\right\} \quad (1.15)$$

This system must be solved under the initial conditions:

$$t = 0; \quad m_1 = N, \quad m_2 = m_3 = m_4 = 0. \quad (1.16)$$

The integration of this system of differential equations for the concrete/specific/actual values of the entering it parameters (N, λ_{12}, λ_{31}, λ_{24}, λ_{34}, λ_{43}) is simplest to carry out in machine or by hand, by numerical integration.

Let us assume that this is realized and you obtained four functions, which express the average numbers of states:

$$m_1(t),\ m_2(t),\ m_3(t),\ m_4(t).$$

Let us find the dispersions of the numbers of states:

$$D_1(t),\ D_2(t),\ D_3(t),\ D_4(t).$$

Earlier we showed that

$$D_k(t)=Np_k(t)\left(1-p_k(t)\right). \qquad (1.17)$$

Hence, considering dependence $m_k(t)=Np_k(t)$, we will obtain:

$$D_k(t)=m_k(t)\left(1-\frac{m_k(t)}{N}\right),\quad k=1,2,3,4. \qquad (1.18)$$

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Thus if the intensities of flow of the events, which translate cell/element from state into state, do not depend on the numbers of states, then, after computing the average numbers of states $m_1(t), \ldots, m_n(t)$, it is possible to immediately find the dispersions of the numbers of states from the formulas:

$$\left.\begin{aligned} D_1(t) &= m_1(t)\left(1-\frac{m_1(t)}{N}\right), \\ D_2(t) &= m_2(t)\left(1-\frac{m_2(t)}{N}\right), \\ &\cdots\cdots\cdots\cdots \\ D_n(t) &= m_n(t)\left(1-\frac{m_n(t)}{N}\right) \end{aligned}\right\} \tag{1.19}$$

and to their middle standard deviations:

$$\sigma_1(t)=\sqrt{D_1(t)};\quad \sigma_2(t)=\sqrt{D_2(t)};\ \ldots;\ \sigma_n(t)=\sqrt{D_n(t)}. \tag{1.20}$$

Let us note that knowing mathematical expectations and the root-mean-square deviation of the numbers of states, we obtain possibility to consider also the probabilities of different states of system as a whole, i.e., for example, the probability of the fact that the number of some state there will be included within certain limits. It is real/actual, let us suppose that the number of cell/elements N in system is great. Then the number of some (the k-th) state can be approximately considered distributed according to normal law. But if this then, then is probability that random variable X_k (number of k state) will be included in some boundaries from α and to β, it will be expressed by the formula:

$$P(X_k \in (\alpha, \beta)) = \Phi\left(\frac{\beta-m_k}{\sigma_k}\right) - \Phi\left(\frac{\alpha-m_k}{\sigma_k}\right). \tag{1.21}$$

where m_k, σ_k — a mathematical expectation and the average quadratic disconnection/cutoff of the number of k state, $\Phi(x)$ - a function of Laplace (see appendix, Table 1).

Let us return to equations for the average numbers of states and will formulate the rule of their composition. It consists of following.

If of system S, which consists of N of uniform cell/elements of the type $\mathcal{E}$, it occurs the Markovian process, moreover is known the graph/count of the states of each cell/element and are shown the intensity λ_{ij} of all flows of the events, which translate cell/element $\mathcal{E}$ from state into state (not depending on the numbers of states), then for the average numbers of states it is possible to comprise differential equations, using the following mnemonic rule:

the derivative of the average number of state is equal to the sum so many canoes, how many arrow/pointers connected with this state; if arrow/pointer is directed from state, term has a sign

"minus", if into state - positive sign. each term is equal to the product of the intensity of flow of events, which translates cell/element on this arrow/pointer, to the average number of that state from which proceeds the arrcw/pcinter.

The comprised for this rule differential equations, in which unknown functions are the average numbers of states, we will call the equations of the dynamics of average.

Example 2. The physical system S consists of N = 200 uniform cell/elements - instruments $\mathscr{E}$. Each of the instruments can be located in one of the two states:

$\mathscr{E}_1$ — is exact,

$\mathscr{E}_2$ — is defective.

The transition of cell/element from state $\mathscr{E}_1$ into state $\mathscr{E}_1$ occurs on by the action of the flow of malfunctions with intensity λ = 2; the mean time of the repair (restoration/reduction) of instrument is equal to $\bar{t}_p = 1/\mu = 1/3$. To comprise the equations of the dynamics of average and to solve them when at the initial moment

all instruments are exact. To depict dependences $m_1(t)$ and $m_2(t)$ on graph. To find and construct on plotted function $\sigma_1(t), \sigma_2(t)$ — root-mean-square deviation of the numbers of states.

Solution. The graph/count of the states of cell/element takes the form, shown on Fig. 6.3. We designate:

m_1 - average number of exact cell/elements at torque/moment t,

m_2 - an average number of defective cell/elements at the same torque/moment.

Equations for the average numbers of states will be:

$$\left.\begin{aligned}\frac{dm_1}{dt} &= -2m_1 + 3m_2,\\ \frac{dm_2}{dt} &= -3m_2 + 2m_1.\end{aligned}\right\} \qquad (1.22)$$

Instead of two equations it is possible to be limited to one, if one considers that for any t

$$m_2 = N - m_1. \qquad (1.23)$$

Substituting (1.23) in first equation (1.22), we will obtain:

$$\frac{dm_1}{dt} = -5m_1 + 3N. \qquad (1.24)$$

Integrating this equation under the initial condition

$$t=0; \quad m_1 = N,$$

we will obtain:

$$m_1(t) = N\left(\frac{3}{5} + \frac{2}{5}e^{-5t}\right). \qquad (1.25)$$

From (1.23) we have:

$$m_2(t) = N - N\left(\frac{3}{5} + \frac{2}{5}e^{-5t}\right) = \frac{2}{5}N(1 - e^{-5t}). \qquad (1.26)$$

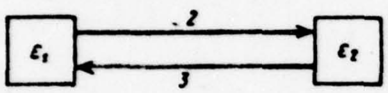

Fig. 6.3.

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Let us construct on plotted function (1.25) and (1.26) (Fig. 6.4). From curve/graph it is evident that with t ⟶ - the average numbers of states approach the limiting values:

$$m_1 \to 3/5N, \quad m_2 \to 2/5N.$$

Let us determine the dispersions of the numbers of states:

$$D_1(t) = m_1(t)\left(1 - \frac{m_1(t)}{N}\right) = N\left(\frac{6}{25} + \frac{4}{25}e^{-5t}\right)(1 - e^{-5t}). \qquad (1.27)$$

It is obvious, the dispersion of the number of second state will be the same:

$$D_2(t) = D_1(t).$$

The root-mean-square deviation of the numbers of states are equal to:

$$\sigma_1(t)=\sigma_2(t)=\sqrt{N\left(\frac{6}{25}+\frac{4}{25}e^{-5t}\right)(1-e^{-5t})}.$$

Plotted function $\sigma_1(t)$ is shown on Fig. to 6.5.

2. Account to the dependence of the intensities of flow of events on the numbers of states. Principle of quasi-regularity.

Until now, applying the method of the dynamics of average, we considered that the intensities of flow of the events, translating cell/element from state into state, to us are previously known and random. Thereby it was assumed that they do not depend on the numbers of states which, as is known, are random. However, in practice very frequently this is not thus. The processes which take place in the system of cell/elements, most frequently store/add up so that the intensities of flow of the events, which translate cell/element from state into state, depend on that, how many cell/elements in this state (yes even in other states) are in system.

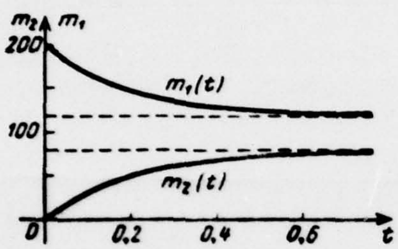

Fig. 6.4.

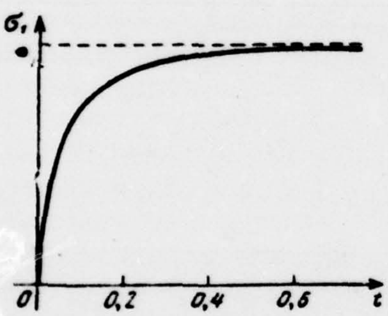

Fig. 6.5.

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For example, in an example of 2 previous paragraphs we assumed that the mean time of the repair of cell/element (value, inversely proportional the intensity of flow of repairs) does not depend on that, how many cell/elements simultaneously are located in repair.

This is real/actual so, if cell/elements so rarely go out of order that virtually cannot be created the "block" during their restoration/reduction. If this not then, it is necessary to consider the fact that the time, required to the repair of cell/element, depends on a quantity of defective cell/elements, available in the presence.

It is real/actual, let us consider system S, which consists of N of uniform cell/elements - instruments which can at random torque/moments go out of order and be directed for repair. let us assume that the repair is realized by one brigade, who has the completely specific capacity (average quantity of repairs per unit time). Then the time which each separate defective cell/element will stay in repair, depends on the total number of overhauled at given torque/moment cell/elements: than this quantity it is more, the greater, on the average, will stay in repair each separate cell/element, and the fact, therefore, will less be the intensity of flow of events, which translates each separate cell/element from state "it is defective" into state "it is exact". Thus, the intensity of flow of events, which translates cell/element from the second state into the first, depends on the number of first state. This number is random - it means the intensity of the translating flow, strictly speaking, it will be random.

By friend an example. Let system S consist of the large number N of the trucks each of which can being able "exact" or "defective". The park/fleet of trucks performs the completely specific circle of works, so that with a large quantity of defective machines the load, which lies down on exact, increases, and, this means, increases the intensity of flow of the events, which translate into them state "is defective". Again the intensity of flow of events depends on number of state.

In the general case (below we will see a series of such examples) the intensities of flow of the events, which translate cell/element from state into state, can depend on the number not of one state, but immediately several. In the case when the intensities of flow of events depend on the number states (that means that are random), we no longer can how this was earlier, write the equations of the dynamics of average, since we do not know the numbers of states, determining intensity. However, this difficulty can be gone around, if one assumes that the intensity of flow of the events, which translate cell/element from state into state they depend not on very numbers of states, but on their average values (mathematical expectations) $m_1, m_2, \ldots, m_n$.

This assumption, which we, following I. Ya. Diner [13], let us call "Principle of quasi-regularity", it will make it possible to

write the equations of the dynamics of average and to solve the task (true, not it is accurate, but approximately, because same this assumption - not precise, but approximated).

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Let us note that the assumption, under discussion, leads to the significant errors only when the total number of cell/elements N in system S comparatively little - then the actual numbers of states they can strongly differ from its mathematical expectations. But if the total number of cell/elements N is great, the deviation of the number of each state from average value is relatively small, and the method of the dynamics of average gives comparatively small errors.

Is essential also the form of dependence, which connects the intensities of flow of events with numbers of states. The nearer this dependence to by the linear (in the range of the virtually possible values of arguments), the lesser the error gives the replacement of random numbers by their average values.

Let us explain the methodology of the use of the principle of quasi-regularity based on examples.

Example 1. System S consists of the large number N of uniform

technical equipment/devices each of which can be in one of the two states:

$\mathcal{E}_1$ — exact, it works,

$\mathcal{E}_2$ — defective, it is overhauled.

On each cell/element functions the flow of malfunctions with the intensity λ, which does not depend on the numbers of states. The repair of cell/elements occupied the group of the working in composition of k persons (k << N). Each defective cell/element is overhauled by one worker (there is no mutual assistance between them); each worker it can overhaul on the average μ of cell/elements per unit time. At the initial moment (t = 0) all cell/elements are correct. All flows of events - Poisson (it can be, with alternating/variable intensity). To write the equations of the dynamics of average for average numbers states.

Solution. The graph/count of the states of cell/element (one technical equipment/device) takes the form, presented in Fig. 6.6, where $\tilde{\mu}$ - the intensity of flow of repairs, which is necessary to one overhauled cell/element.

Let us find the dependence $\tilde{\mu}$ on number X_2 of cell/elements,

which are found at given torque/moment in the state of repair. Let us begin from the fact that let us determine, with the datum X_2, the total intensity M_Σ of the flow of repairs, which is necessary to all cell/elements which are located in state s_2. This total intensity there is a function of the number of cell/elements, which are found in the state of the repair:

$$M_\Sigma = \varphi(X_2).$$

Since workers work without mutual assistance and number their is equal to k, then the total intensity of flow of repairs with the increase of the number of overhauled cell/elements increases according to linear law (of proportionally the number overhauled cell/elements) until their number is achieved k; after this all workers will be occupied, intensity M_Σ will cease to increase and will remain equal to μk:

$$M_\Sigma = \varphi(X_2) = \begin{cases} \mu X_2 & \text{при } X_2 \leqslant k, \\ \mu k & \text{при } X_2 > k. \end{cases} \quad (2.1)$$

Key: (1). with.

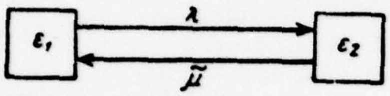

Fig. 6.6.

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Let us construct plotted function $\phi(X_2)$ (see Fig. 6.7). It is assigned only at integral points; but during the composition of the equations of the dynamics of average with the use of principle of quasi-regularity for us it is necessary to replace random number X_2 of cell/elements in the state of repair to its mathematical expectation m_2, but it can be whole. Therefore we should determine function ϕ, also, for the nonone-piece/entire values of argument. For this, we will use linear interpolation and will connect points on the graph of Fig. 6.7 by line segments.

Let us count now, which will be the average intensity of flow of repairs, which is necessary to one overhauled cell/element:

$$\tilde{\mu}=\varphi_1(X_2)=\frac{M_\Sigma}{X_2}.$$

Further (2.1) from X_2, we will obtain:

$$\tilde{\mu} = \varphi_1(X_2) = \begin{cases} \mu & \text{при } X_2 \leqslant k, \\ \dfrac{\mu k}{X_2} & \text{при } X_2 \geqslant k. \end{cases} \qquad (2.2)$$

Key: (1). with. Plotted function $\phi_1(X_2)$ is represented in Fig. 6.8. This curve, as $\phi(X_2)$, consists of two sections. On the first (from 0 to k) it is parallel to the axis of abscissas, on the second - decreases according to hyperbolic law.

Now to us is known the intensity of flow of events $\lambda_{21} = \tilde{\mu}$, translating one cell/element of state $\mathcal{E}_2$ in $\mathcal{E}_1$. It depends on the actual (random) number X_2 of cell/elements, which are found in state $\mathcal{E}_2$. According to the principle of quasi-regularity, let us replace this random number with its mathematical expectation m_2. Then, with the basis of the graph/count of states (Fig. 6.6), the differential equations of the dynamics of average are registered in the form:

$$\frac{dm_1}{dt} = -\lambda m_1 + \varphi_1(m_2)\, m_2, \qquad (2.3)$$

$$\frac{dm_2}{dt} = -\varphi_1(m_2)\, m_2 + \lambda m_1, \qquad (2.4)$$

where m_1, m_2 - average numbers of states $\mathcal{E}_1$, $\mathcal{E}_2$.

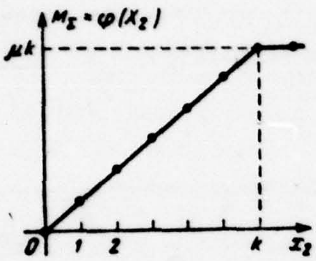

Fig. 6.7.

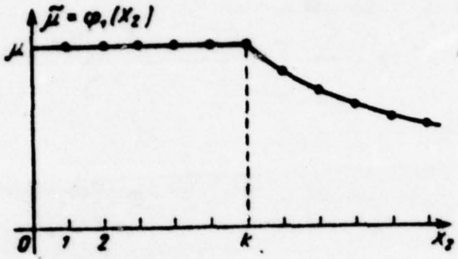

Fig. 6.8.

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Equations (2.3), (2.4) can be rewritten in other form, if we remember that

$$\varphi_1(X_2) = \frac{\varphi(X_2)}{X_2}; \quad \varphi_1(m_2) = \frac{\varphi(m_2)}{m_2}.$$

We will obtain two equations:

$$\left.\begin{aligned} \frac{dm_1}{dt} &= -\lambda m_1 + \varphi(m_2), \\ \frac{dm_2}{dt} &= -\varphi(m_2) + \lambda m_1. \end{aligned}\right\} \quad (2.5)$$

From these two equaticns we can select one - for example, the second, first reject/throw and in the second substitute expression m_1 of the condition:

$$m_1+m_2=N;\quad m_1=N-m_2.$$

We will obtain instead of (2.5) one the differential equation:

$$\frac{dm_2}{dt}=-\varphi(m_2)+\lambda(N-m_2).$$

This - equation from separating alternating/variable:

$$\frac{dm_2}{\lambda(N-m_2)-\varphi(m_2)}=dt.$$

Integrating right side from 0 to t, and a left - from 0 to m_2 (initial value of m_2 is equal to zero), we have:

$$\int_0^{m_2}\frac{dm_2}{\lambda(N-m_2)-\varphi(m_2)}=t. \qquad (2.6)$$

Taking into account that the function $\varphi(m_2)$ is assigned by two different expressions with $m_2 \leqslant k$ and with $m_2 < k$, we have:

with $m_2 \leqslant k$

$$t=\int_0^{m_2}\frac{dm_2}{\lambda(N-m_2)-\mu m_2}=\int_0^{m_2}\frac{dm_2}{\lambda N-(\lambda+\mu)m_2}=$$
$$=-\frac{1}{\lambda+\mu}\ln\frac{\lambda N-(\lambda+\mu)m_2}{\lambda N},$$

whence

$$m_2=\frac{\lambda N}{\lambda+\mu}[1-e^{-(\lambda+\mu)t}]. \qquad (2.7)$$

With $m_2 > k$

$$t=\int_0^{k}\frac{dm_2}{\lambda N-(\lambda+\mu)m_2}+\int_k^{m_2}\frac{dm_2}{\lambda N-k\mu-\lambda m_2}.$$

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The first integral is equal to:

$$-\frac{1}{\lambda+\mu}\ln\frac{\lambda N-(\lambda+\mu)k}{\lambda N}.$$

We compute the second integral:

$$\int_k^{m_2}\frac{dm_2}{\lambda N-k\mu-\lambda m_2}=-\frac{1}{\lambda}\ln(\lambda N-k\mu-\lambda m_2)\Big|_k^{m_2}=$$
$$=-\frac{1}{\lambda}\ln\frac{\lambda N-k\mu-\lambda m_2}{\lambda N-k(\lambda+\mu)}.$$

Consequently, with $m_2 > k$

$$t=-\frac{1}{\lambda+\mu}\ln\frac{\lambda N-(\lambda+\mu)k}{\lambda N}-\frac{1}{\lambda}\ln\frac{\lambda N-k\mu-\lambda m_2}{\lambda N-k(\lambda+\mu)}.$$

whence

$$m_2 = N - k\frac{\mu}{\lambda} - \frac{N - k\,\dfrac{\lambda+\mu}{\lambda}}{\left[1 - \dfrac{\lambda+\mu}{\lambda N}k\right]^{\lambda/(\lambda+\mu)}}\,e^{-\lambda t}. \qquad (2.8)$$

By formula (2.7) value m_2 will be expressed with

$$t \leqslant -\frac{1}{\lambda+\mu}\ln\frac{\lambda N-(\lambda+\mu)k}{\lambda N} = \frac{1}{\lambda+\mu}\ln\frac{\lambda N}{\lambda N-(\lambda+\mu)k},$$

while by formula (2.8) - at the high values of t.

Example 2. The conditions the same as in example 1, with that difference that k workers, which overhaul the left the system cell/elements, help each other, so that k workers realize a repair of one cell/element on the average in k of times sooner than one worker.

It is required to construct for these conditions function $M_\Sigma = \varphi(X_2)$ (total intensity of flow of repairs), function $\mu = \varphi_1(X_2)$ (intensity of flow of repairs, which is necessary to one overhauled cell/element) and to comprise differential equations for the average numbers of states (equation of the dynamics of average).

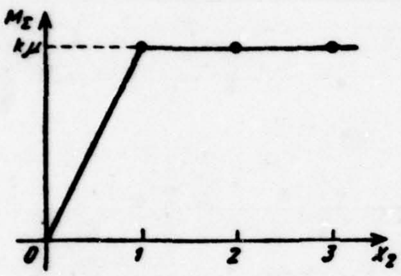

Fig. 6.9.

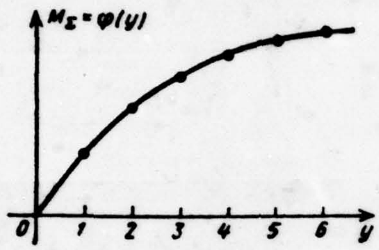

Fig. 6.10.

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Solution. The graph/diagram of the dependence M_Σ on the number of overhauled cell/elements X_2 is represented in Fig. 6.9. It is real/actual, with any positive integer number of cell/elements, which are found in state s_1 (X_2 = 1, 2, 3,...), all workers, working simultaneously above the repair of these cell/elements, they generate one flow of repairs with an intensity of $k\mu$; they as are equivalent to one "super-worker" with productivity, in k of times larger (see 9 Chapter 5). In such a way as to solve stated problem, it suffices

under conditions of example 1 to place k = 1, and instead of μ - to substitute $k\mu$.

Under actual conditions the dependence of the total intensity of flow of repairs M_{Σ}, on the number of overhauled cell/elements can be and not such an idle time, as in the examined two examples - it can depend on the special feature/peculiarities of the organization of repairs in brigade, on the order of the maintenance of cell/elements on the capacitance/capacity of repair shops, and so forth. It can turn out that for the establishment of the form of functions $M_{\Sigma} = \varphi(X_{\Sigma})$ it is necessary to make the special research, for example, considering repair team as system of mass maintenance and system for it mathematical model.

Example 3. Is examined the system, which consists of N = 100 identical instruments; each instrument consists of two identical assemblies: one basic, the second spare. In the case of breakdown of basic assembly in work, is included spare. During the malfunction of both assemblies, goes out of order and ceases to work entire instrument. The flow of malfunctions, which functions on working assembly, has intensity λ_1; to that not work (exact) - λ_2. Malfunctioning assemblies are overhauled by working team. The total intensity of flow of brigade's repairs, depending on the total number of overhauled assemblies y, is assigned by the function

$$M_{\Sigma} = \varphi(y).$$

The form of the function $\varphi(y)$ is represented in Fig. to 6.10.

Separate instrument (cell/element) can be located in the following states:

$\mathcal{E}_1$ — exact both assembly, the first works, the second in re erve,

$\mathcal{E}_2$ — the first assembly is defective, is overhauled, the second assembly works; instrument works,

$\mathcal{E}_3$ — both assembly are defective, they are overhauled; instrument does not work.

The left the system units are overhauled regardless of the fact, is unit basic or spare (repairs are distributed on units evenly). After the correction of the left the system unit, it becomes spare if another did not leave the system, and by basic - if left.

To write the equations of the dynamics of average.

Solution. The graph/count of the states of cell/element (instrument) takes the form, shown on Fig. 6.11.

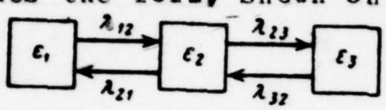

Fig. 6.11.

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Let us determine the intensity of the flows of the events, which translate cell/element from state into state. First of all,

$$\lambda_{12} = \lambda_1 + \lambda_2.$$

It is real/actual, thus far instrument works normally, on both unit, function the flows of the malfunctions: to worker - with intensity λ_1, to that not work - with intensity λ_2. To instrument as a whole, functions the flow with total intensity $\lambda_1 + \lambda_2$.

Further, from state $\mathcal{E}_2$ into $\mathcal{E}_3$ instrument passes under the action of the flow of malfunctions, which is necessary on the only working unit:

$$\lambda_{23} = \lambda_1.$$

Conversely, from state $\mathcal{E}_2$ into $\mathcal{E}_1$, instrument is translated the

flow of repairs, which is necessary to one overhauled unit. The total number of units, that is located in repair, is equal

$$y = X_2 + 2X_3.$$

It is real/actual, to each instrument, which is found in state $\mathscr{E}_2$, with is gone one defective unit; to each instrument in state $\mathscr{E}_3$— two defective units.

The total intensity of flow of repairs will be:

$$M_\Sigma = \varphi(y) = \varphi(X_2 + 2X_3).$$

This intensity is divided equally between all overhauled units, so that to one unit comes the intensity of flow of repairs, equal to

$$\tilde{\mu} = \frac{\varphi(X_2 + 2X_3)}{X_2 + 2X_3}.$$

Consequently, the true intensity of flow of repairs, which is necessary to one cell/element in state $\mathscr{E}_2$, is equal to:

$$\lambda_{21} = \frac{\varphi(X_2 + 2X_3)}{X_2 + 2X_3}.$$

Analogously let us determine λ_{32}. In state $\mathscr{E}_3$ the instrument has two defective units; for each of them, comes the flow of repairs

with the intensity

$$\frac{\varphi(X_2+2X_3)}{X_2+2X_3}.$$

and to both - flow with intensity, dcuble larger:

$$\frac{2\varphi(X_2+2X_3)}{X_2+2X_3}.$$

According to the principle of quasi-regularity, we replace the random arguments X_2 and X_3 with the mathematical expectations m_2 and m_3; we will obtain:

$$\lambda_{31}\approx\frac{\varphi(m_2+2m_3)}{m_2+2m_3};\quad \lambda_{32}\approx\frac{2\varphi(m_2+2m_3)}{m_2+2m_3}.$$

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Thus, it is possible to write on the graph/count of states all the intensities and, according to general rule, to register the equations of the dynamics of average. Of three equations (for m_1, m_2 and m_3) we write the first and the latter - the second we reject/throw:

$$\left.\begin{aligned}\frac{dm_1}{dt}&=-(\lambda_1+\lambda_3)m_1+\frac{\varphi(m_2+2m_3)m_2}{m_2+2m_3},\\ \frac{dm_3}{dt}&=-\frac{2\varphi(m_2+2m_3)m_3}{m_2+2m_3}+\lambda_1m_2.\end{aligned}\right\}\qquad(2.9)$$

From the condition

$$m_1+m_2+m_3=N$$

we express m_2 through m_1 and m_3:

$$m_2=N-m_1-m_3$$

we substitute in first and second equations (2.9):

$$\left.\begin{aligned}\frac{dm_1}{dt}&=-(\lambda_1+\lambda_2)\,m_1+\frac{\varphi\,(N-m_1+m_3)\,(N-m_1-m_3)}{N-m_1+m_3}.\\ \frac{dm_3}{dt}&=-\frac{2\varphi\,(N-m_1+m_3)\,m_3}{N-m_1+m_3}+\lambda_1\,(N-m_1-m_3).\end{aligned}\right\}\quad(2.10)$$

The obtained system of two nonlinear differential equations with the unknown functions m_1, m_3 can be solved in machine or by hand (it is numerical).

Thus, using the principle of quasi-regularity, it is possible to write the equations of the dynamics of the average, on which unknown functions are the average numbers of state; these equations approximately describe a change of the average numbers of states even in the case when the intensity of flow of the events, which translate cell/element from state into state, they depend on the numbers of

states and, which means, that they are random. The error from which the equations of the dyramics of average describe process, the lesser than more it is numerous the group of cell/elements and than nearer to linear the functions, which express the intensity of flow of events depending on the numbers of states.

Does arise the question: a it is cannot whether, using the same method, that in §1, to approximately determine not only mathematical expectations, but also the dispersions of the average numbers of states? We saw that in the case when separate cell/elements passed from state into state independent from each other (i.e. the intensities of flow of the events, translating cell/elements from the state in the state completely did not depend on the numbers of states) the dispersions of the numbers of states were located simply through the formula:

$$D_k(t) = m_k(t)\left(1 - \frac{m_k(t)}{N}\right). \qquad (2.11)$$

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Research shows that in the case when the intensities of flow of events depend on the numbers of states, this formula, generally speaking, cannot be used. It proves to be suitable only in cases when dependence the intensity of flow of events on numbers very weak

(almost negligible), even then on comparatively low sections of time, when did not accumulate error. But if the dependence of intensities on numbers is essential, formula (2.11) gives error. If the functions, which express total intensities of flow (as, for instance, function ϕ in example 1) are convex upward, then formula (2.11) gives decreased value for the dispersion: the dispersion, computed on this formula, can be let us say, that half true (but sometimes - and more than double).

To approximately find the dispersions of the numbers of states is possible by writing out and by solving special differential equations no longer for mathematical expectations, but for dispersions $D_k(t)$ and covariances $K_{kl}(t)$, the characterizing communication/connection between numbers states $\mathscr{E}_k, \mathscr{E}_l$. These equations in some sense are analogous to the equations of the dynamics of average, comprised on the basis of the principle of quasi-regularity, but much more complex them and they do not possess the same clarity. The number of equations and the number of unknowns in these equations is equal to the number of dispersions plus the number of pairwise correlations between numbers X_1, X_2, ..., X_n, i.e.

$$n+\frac{n(n-1)}{2}=\frac{n(n+1)}{2}.$$

Besides dispersions $D_k(t)$ and the correlation torque/moments

$K_{kl}(t)$, into equations for them enter even n of functions $m_1(t)$, $m_2(t)$, ..., $m_n(t)$ — average numbers of states which are assumed to be already determined from the equations of the dynamics of average. equations for dispersions $D_k(t)$ and covariances $K_{kl}(t)$ prove to be relative to quite these variables linear, although the mathematical expectations of numbers $m_k(t)$ enter in them nonlinear.

In view of a comparative complexity of a question, we do not examine the methodology of the construction of system of equations for dispersions and covariances (for a special case of equation are described in article [22]).

3. Account of the addition/completion of the numbers of states.

Until now, we applied the method of the dynamics of average to the decision only of such tasks where the system was locked, i.e., a quantity of cell/elements N, participating in process, remained constant/invariable. In practice frequently are encountered the tasks where in the course of the process of the number of cell/elements, which are found in some states, they are filled from without. This addition/completion it is very easy to take into account into the equations of the dynamics of average.

Let us consider as an example system S, which consists of N of

uniform cell/elements. The graph/count of the states of cell/element is shown in Fig. 6.12.

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Intensities λ_{ij} in the general case depend on the numbers of states X_1, X_2, X_3, X_4 (during the composition of differential equations these numbers are substituted by the average numbers m_1, m_2, m_3, m_4).

If the addition/completion of the composition of the numbers of states in the course of process does not occur, then the equations of the dynamics of average will be:

$$\left.\begin{aligned}
\frac{dm_1}{dt} &= -(\lambda_{12}+\lambda_{13})\,m_1+\lambda_{21}\,m_2,\\
\frac{dm_2}{dt} &= -(\lambda_{21}+\lambda_{23}+\lambda_{24})\,m_2+\lambda_{12}\,m_1,\\
\frac{dm_3}{dt} &= \lambda_{13}\,m_1+\lambda_{23}\,m_2+\lambda_{43}\,m_4,\\
\frac{dm_4}{dt} &= -\lambda_{43}\,m_4+\lambda_{24}\,m_2,
\end{aligned}\right\} \qquad (3.1)$$

moreover any of these equations of them can be reject/thrown, and the corresponding variable it is expressed from the condition:

$$m_1+m_2+m_3+m_4=N. \qquad (3.2)$$

Now let us assume that the contingent of the cell/elements,

located in one of the states (for example, $\mathscr{E}_1$) is supplemented from without, moreover the intensity of addition/completion, i.e., the number of cell/elements, introduced per unit time into state $\mathscr{E}_1$, is equal to δ (if for time unit it is introduced the random number of units, the intensity of addition/completion it will be called the average number of units, introduced from without for the unit of time). Value δ can be both the constant and variable both depending and not depending on the average numbers of states.

In the presence of addition/completion, the first equation of system (3.1) will be changed; in the right part of it will appear a term equal to addition/completion δ:

$$\frac{dm_1}{dt} = -(\lambda_{12} + \lambda_{13})\, m_1 + \lambda_{21}\, m_2 + \delta, \qquad (3.3)$$

and remaining equations will remain similar, as they were.

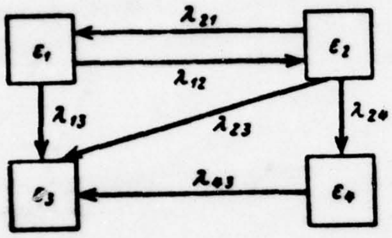

Fig. 6.12.

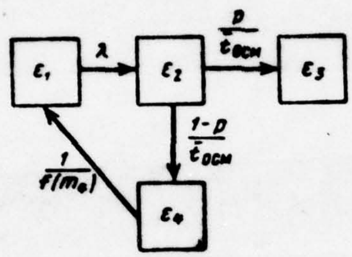

Fig. 6.13.

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Let us note that condition (3.2) also will be changed. Earlier of the any moment of time the sum of all average numbers was equal one and the same to value N; now it will be equal to the changing in the course of time number

$$N(t) = N_0 + \int_0^t \delta(t)\, dt, \quad (3.4)$$

where N_0 - initial value of the number of cell/elements.

Thus, the account of filling of the numbers of states is reduced to the fact that to the right side of the corresponding differential equation is added component, it is equal to the intensity of addition/completion - to the mean number of cell/elements, introduced into this state for time unit.

Example 1. Is considered a system, which consists (at the initial moment) of N_0 of uniform technical equipment/devices (instruments), each of which can be in one of the following states:

$\mathcal{E}_1$ — correct;

$\mathcal{E}_2$ — defective, it will be scanned;

$\mathcal{E}_3$ — considered unfit, it is copied;

$\mathcal{E}_4$ — repaired.

Corresponding average numbers we will designate m_1, m_2, m_3, m_4. The graph/count of the states of cell/element is shown on Fig. 6.13.

The intensity of flow of the malfunctions of working instrument is equal to λ. The mean time of inspection does not depend on the

number of scanned instruments and is equal to $\bar{t}_{осм}$. Defective instrument proves to be unfit and is copied with probability p, but with probability 1 - p, it is directed for repair. The mean time which instrument is carried out in the state of repair, $\bar{t}_{рем}$ there is certain function from number x of the instruments, simultaneously located in the repair:

$$\bar{t}_{рем} = f(x).$$

In order to compensate for the loss/depreciation of instruments as a result of writing off, is produced the addition/completion of the number of instruments from without (by corrective instruments), moreover for the time unit into system it is introduced on the average $\delta = \delta(t)$ of exact instruments.

It is required:

- to write the equations of the dynamics of average taking into account addition/completion,

- to determine, what must be function $\delta(t)$ for, the writing off of instruments on the average would be being compensated for,

- to write formula for the total number of cell/elements N(t),

that are located in all states up to torque/moment t.

Solution. On the graph/count of Fig. 6.13, we enter/write the intensities of flow of events. Intensity λ_{41} we approximately take inversely proportional to the mean time of repair (strictly speaking, this correctly only for a stationary Poisson flow):

$$\lambda_{41} = \frac{1}{\bar{t}(X_4)}$$

Substituting the true number of overhauled instruments X_4 by its mathematical expectation m_4, we will obtain:

$$\lambda_{\vartheta 1} \approx \frac{1}{f(m_4)}.$$

The system of the differential equations of the dynamics of average will be:

$$\left.\begin{aligned} \frac{dm_1}{dt} &= -\lambda m_1 + \frac{m_4}{f(m_4)} + \delta, \\ \frac{dm_2}{dt} &= -\frac{m_2}{\bar{t}_{\text{осм}}} + \lambda m_1, \\ \frac{dm_3}{dt} &= \frac{pm_2}{\bar{t}_{\text{осм}}}, \\ \frac{dm_4}{dt} &= -\frac{m_4}{f(m_4)} + \frac{(1-p)\,m_2}{\bar{t}_{\text{осм}}} \end{aligned}\right\}$$

Let us note that in this case we not can so simple to reject/throw any of the equations as in the case without addition/completion, since condition (3.2) is modified; the total number of cell/elements in system depends on time and it is equal to:

$$N(t) = N_0 + \int_0^t \delta(t)\,dt. \qquad (3.5)$$

In order on the average to compensate for the copied instruments, the intensity of addition/completion must be equal to the average number of instruments, copied for time unit. In all per

unit time it is copied (it passes from state $\mathcal{S}_2$ in $\mathcal{S}_3$) on the average

$$\frac{p}{\bar{t}_{\text{осм}}} m_2$$

instruments; that means we must assume:

$$\delta = \frac{p}{\bar{t}_{\text{осм}}} m_2.$$

At this intensity of addition/completion, the system of equations of the dynamics of average will take the form:

$$\left.\begin{aligned}
\frac{dm_1}{dt} &= -\lambda m_1 + \frac{1}{f(m_4)} m_4 + \frac{p}{\bar{t}_{\text{осм}}} m_2,\\
\frac{dm_2}{dt} &= -\frac{1}{\bar{t}_{\text{осм}}} m_2 + \lambda m_1,\\
\frac{dm_3}{dt} &= \frac{p}{\bar{t}_{\text{осм}}} m_2,\\
\frac{dm_4}{dt} &= -\frac{1}{f(m_4)} m_4 + \frac{1-p}{\bar{t}_{\text{осм}}} m_2.
\end{aligned}\right\} \qquad (3.6)$$

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From the number of equations (3.6) it is possible to painless exclude the third, since value m_3 enters not in one right side. Value m_3 under conditions of this example can be computed very simply: for each torque/moment t, it is equal to the total number of newly acted instruments (since everything copied on the average are compensated for) and, which means, that

$$m_3 = \frac{p}{\bar{t}_{\text{осм}}} \int_0 m_2(t)\, dt.$$

In this example 1 addition/completion was introduced only into one state; generally, this can be and not so (for example, it is possible to introduce addition/completion by the defective instruments which must be overhauled by local resources). Let us note that besides the fact that the functions of addition/completion can have both positive and negative values (loss/depreciation of cell/elements).

The addition/completion, introduced into states, sometimes it is convenient to represent visually, on the graph/count of states (Fig. 6.14). Let us agree to represent them as " half-arrows" which do not go either from which state, and in the case of "loss/depreciation" - those directed or to which state (for the clarity of half-arrows, unlike arrow/pointers, let us make double). Labeling graph/count with the intensities of flow of events, against half-arrows let us write not the intensity, which is necessary to one cell/element, but intensity being necessary to system as a whole (this is made in order to avoid unnecessary division and multiplication by one and the same number).

Example of 2. under conditions of an example 1 addition/completion of numbers is related to two states: S_1 (exact instruments) and S_4 (overhauled instruments), moreover the certain fraction α of the newly supplied instruments is given exact, and

fraction $(1 - \alpha)$ - defective; the latter immediately begin to be overhauled. As in the previous example, total addition/completion per unit time is equal to $\frac{p}{t_{ocm}} m_s$.

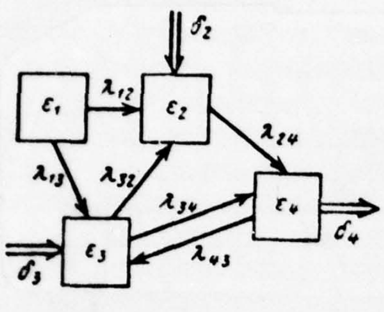

Fig. 6.14.

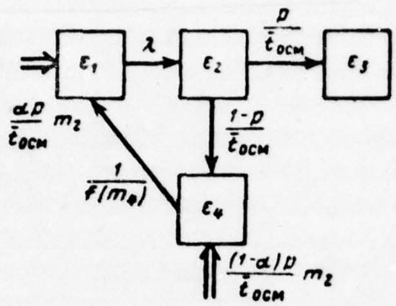

Fig. 6.15.

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To construct the graph/count of states, after reflecting in it addition/completion, to write the equations of the dynamics of average, to define grand average quantity of cell/elements in system N(t) as function of time.

Solution. The graph/count of states is shown on Fig. 6.15; half-arrows, directed to states $\mathscr{E}_1$, $\mathscr{E}_4$, represent addition/completion. The equations of the dynamics of average take the form:

$$\left.\begin{aligned}
\frac{dm_1}{dt} &= -\lambda m_1 + \frac{1}{f(m_4)} m_4 + \frac{\alpha p}{\bar{t}_{осм}} m_2,\\
\frac{dm_2}{dt} &= -\frac{1}{\bar{t}_{осм}} m_2 + \lambda m_1,\\
\frac{dm_3}{dt} &= \frac{p}{\bar{t}_{осм}} m_2,\\
\frac{dm_4}{dt} &= -\frac{1}{f(m_4)} m_4 + \frac{1-p}{\bar{t}_{осм}} m_2 + \frac{(1-\alpha)p}{\bar{t}_{осм}} m_2.
\end{aligned}\right\} \quad (3.7)$$

From them is most convenient as before to exclude the third equation and to express m_3 as

$$m_3 = \frac{p}{\bar{t}_{ocm}} \int_0^t m_2\, dt.$$

The total total number of cell/elements in system varies in time according to the formula:

$$N(t) = N_0 + \frac{p}{\bar{t}_{ocm}} \int_0^t m_2 dt = N_0 + m_3.$$

4. Method of the dynamics of average for the system, which consists of heterogeneous cell/elements.

Until now, we applied the method of the dynamics of average to the systems, consisting of uniform cell/elements. However, without fundamental changes, it can be used also to the systems, which consist of the heterogeneous cell/elements of different categories - a difference will be only in the fact that the number of differential equations will be increased. If the number of categories and states not is too great, the solution of problem difficulties is not caused.

Example. In motor transport establishment is $N^г$ cargo and $N^п$ passenger trucks. Each cargo machine can be in one of the following states: $Г_1$ - expects call on basis, $Г_2$ — accomplishes empty voyage

to the place loads Γ_3 — accomplishes vcyage with load, Γ_4 — accomplishes empty voyage back to base, Γ_5 — conduct the preventive inspection, Γ_6 — is overhauled.

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Each passenger machine can be in cne cf from the following states: $Л_1$ — expects call cn basis, $Л_2$ — accomplishes voyage, $Л_3$ — it passes the preventive inspection, $Л_4$ — is overhauled.

The basis enter the Pcisson flows of claims for the cargo and passenger machines whose intensities $\lambda^г$ and $\lambda^п$ do not depend on the number of free machines on basis. The ccme claims are distributed evenly between all machines of this category, which expect call. If on basis there is not cne free machine cf this category, claim obtains the failure (it will be re-addressed tc another basis).

To the preventive inspection are taken only the machines, which are found in states Γ_1, $Л_1$. The average intensity of flow of the preventive inspections cf cargc machine is equal to $\lambda^г_{проф}$, passenger $\lambda^п_{проф}$.

Inspections are carried out by the specialized brigade; the total flow of inspections has the intensity

$$A_\Sigma = a(1 - e^{-\nu}), \qquad (4.1)$$

where y - a number of machines (cargo and passenger together), that pass inspection.

The average duration of the preventive inspection of cargo and passenger machine is identical and equal to $\bar{t}_{осм}$. The average duration of empty voyage (to the loading site or to motor depot) is equal to $\bar{t}^{г}_{пор}$. The average duration of loaded voyage is equal to $\bar{t}^{г}_{гр}$. The average duration of the voyage of passenger machine is equal to $\bar{t}^{п}$.

From the preventive inspection cargo machine with probability $p^{г}$ goes into repair, and with probability 1 - $p^{г}$ - back into state Γ_1. Analogous probabilities for passenger machines are equal to $p^{п}$ and 1 - $p^{п}$.

The repair both of cargo and passenger machines is produced by maintenance crew; the total flow of repairs, produced by brigade, has the intensity

$$B_{\Sigma} = b(1 - e^{-x}), \qquad (4.2)$$

where x - a number of machines (cargo and passenger together), simultaneously locating in repair.

Besides the state of the preventive inspection, machine, they can enter repair directly from voyage. The intensity of flow of the

malfunctions of one cargo machine in the state of empty voyage is equal to v^{r}_{nop}, in the state of loaded voyage, - v^{r}_{rp}. The intensity of flow of the malfunctions of passenger machine, which is located in voyage, is equal to v^{n}.

It is required: - to comprise the graph/count of the states of the elements of system, to write differential equations for the average numbers of states.

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Solution. It is introduced the designation:

m_1^r — average number of cargo machines, which expect call at torque/moment t;

m_2^r — average number of cargo machines, which accomplish empty voyage to the loading site;

m_3^r — average number of cargo machines, which accomplish loaded voyage;

m_4^r — average number of cargo machines, which are returned with empty car to basis;

m_5^r — average number of cargo machines, which pass the preventive inspection;

m_6^r — average number of overhauled cargo machines;

m_1^n — average number of passenger machines, which expect call;

m_2^n — average number of passenger machines, which accomplish voyage;

m_3^n — average number of passenger machines, which pass the preventive inspection;

m_4^n — average number of overhauled passenger machines.

The graph/count of the states of system, who falls into two subgraph - Г and Л, is shown on Fig. 6.16.

Let us determine now intensities $\lambda_{ij}^r, \lambda_{ij}^n$ the flows of the events, which translate cell/elements (cargo and passenger machines) from state into state. Some of these intensities depend on the numbers of states, others dc not depend. For the first we, during the

composition of differential equations, according to the principle of quasi-regularity, will replace for the number of states on which they depend, by average numbers.

Let us find λ_{12}^{Γ} - intensity of flow of events, which translates the cargo machine, which expects call, in state Γ_2 - voyage to the loading site. The calls of cargo machines, according to the conditions of problem, form flow with intensity λ^{Γ}; but call is received only if in state Γ_1 there is at least one machine.

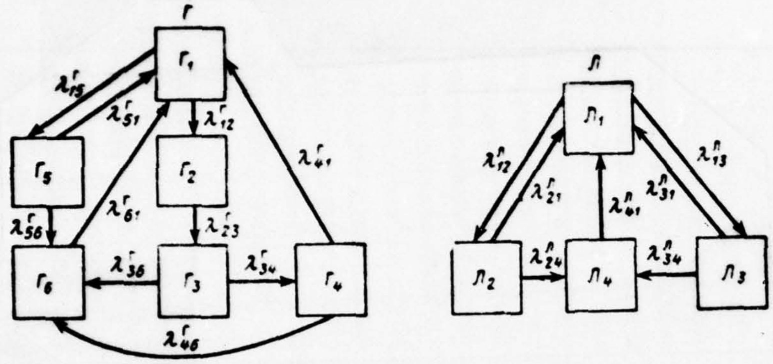

Fig. 6.16.

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Therefore intensity of the flow of received calls $\lambda^{г}_{пр}$ (but only such calls they can translate cargo machine from state $Г_1$ in $Г_2$) depends on the number $X^{г}_1$ of machines in state $Г_1$ as follows:

$$\lambda^{г}_{пр}=\begin{cases}\lambda^{г} \text{ with } X_1{}^{г}>1,\\ 0 \quad \text{при } X_1{}^{г}=0.\end{cases} \tag{4.3}$$

This function of the same form, as to us it was encountered already earlier, in example 2 §2, and will be met still repeatedly in following examples. Therefore we will now introduce two functions which subsequently will be designated always equally: R(x) and ρ(x) (them we will use in many specific problems of the dynamics of average).

Let us determine function R(x) as follows:

$$R(x)=\begin{cases} x \text{ with } x\leqslant 1; \\ 1 \text{ при } x>1. \end{cases} \tag{4.4}$$

The graph of this function is represented in Fig. to 6.17.

Function $\rho(x)$ let us determine formula:

with

$$\rho(x)=\frac{R(x)}{x}=\begin{cases} 1 \text{ при } x\leqslant 1, \\ \frac{1}{x} \text{ при } x>1. \end{cases} \tag{4.5}$$

The graph of function $\rho(x)$ is given on Fig. 6.18.

With the help of function R(x) the intensity $\lambda^{\Gamma}_{\text{пр}}$ of the flow of the calls of cargo machines accepted is record/written as follows:

$$\lambda^{\Gamma}_{\text{пр}}=\lambda^{\Gamma} R(X^{\Gamma}_{1}). \tag{4.6}$$

Let us now compute the intensity λ^{Γ}_{12} of the flow of the events, which translate separate cargo machine from state Γ_1 in Γ_2 (see Fig. 6.16):

$$\lambda_{12}{}^{\Gamma}=\frac{\lambda^{\Gamma}_{\text{пр}}}{X_1{}^{\Gamma}}=\lambda^{\Gamma}\frac{R(X_1{}^{\Gamma})}{X_1{}^{\Gamma}}=\lambda^{\Gamma}\rho(X_1{}^{\Gamma}). \tag{4.7}$$

Let us further find other intensities. We have:

$$\lambda^{\Gamma}_{23}=1/\bar{t}^{\Gamma}_{\text{пор}};\quad \lambda^{\Gamma}_{34}=1/\bar{t}^{\Gamma}_{\text{гр}};\quad \lambda^{\Gamma}_{41}=1/\bar{t}^{\Gamma}_{\text{пор}};\quad \lambda^{\Gamma}_{15}=\lambda^{\Gamma}_{\text{проф}}. \tag{4.8}$$

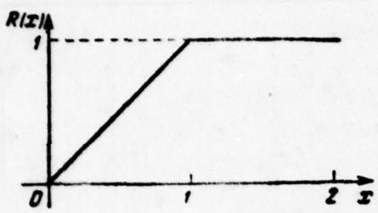

Fig. 6.17.

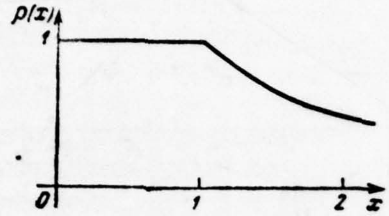

Fig. 6.18.

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Let us now determine the intensity λ^{Γ}_{51} of the flow of events, which translates cell/element (cargo machine), which is found in state Γ_5 (the preventive inspection), into state Γ_1. This intensity let us compute as follows. The total intensity of flow of inspections which produces the brigade, according to formula (4.1), is equal to:

$$A_{\Sigma}=a\left[1-e^{-(X_5{}^{\Gamma}+X_3{}^{Л})}\right]. \qquad (4.9)$$

This intensity must be equally divided between all machines, which are found in states Γ_5 and $Л_3$; it will be obtained

$$\frac{a\left[1-e^{-(X_5{}^{\Gamma}+X_3{}^{Л})}\right]}{X_5{}^{\Gamma}+X_3{}^{Л}} \text{ (inspections per unit time)}. \qquad (4.10)$$

But this still not all: indeed into state Γ_1 passes each machine, passed inspection, but only some part of them. In order to

obtain from (4.10) intensity λ_{51}, it is necessary to multiply (4.10) by $(1-p^{r})$ - probability that the machine from maintenance will return to state Γ_1; we will obtain:

$$\lambda^{r}_{51}=\frac{a(1-p^{r})\left[1-e^{-(X_5^{r}+X_3^{n})}\right]}{X_5^{r}+X_3^{n}}. \qquad (4.11)$$

Analogously we find:

$$\lambda^{r}_{56}=\frac{ap^{r}\left[1-e^{-(X_5^{r}+X_3^{n})}\right]}{X_5^{r}+X_3^{n}}. \qquad (4.12)$$

The intensity of flow of the repairs, which translate cell/element from state Γ_6 in Γ_1, will be expressed by the formula:

$$\lambda^{r}_{61}=\frac{b\left[1-e^{-(X_6^{r}+X_4^{n})}\right]}{X_6^{r}+X_4^{n}}. \qquad (4.13)$$

where X_6^{r}, X_4^{n} - number of overhauled in this time cargo and passenger machines.

From states $\Gamma_2, \Gamma_3, \Gamma_4$ into state Γ_6 (repair) the cell/element is translated by the flows of events with the intensities, respectively equal to:

$$\lambda^{r}_{26}=\nu^{r}_{\text{пор}};\quad \lambda^{r}_{36}=\nu^{r}_{\text{гр}};\quad \lambda^{r}_{46}=\nu^{r}_{\text{пор}}. \qquad (4.14)$$

Analogously we determine the intensity of flow of events for the second subgraph (passenger machines):

$$\lambda^{n}_{12}=\lambda^{n}\rho(X_1^{n}), \qquad (4.15)$$

$$\lambda^{n}_{21}=\frac{1}{t^{n}}. \qquad (4.16)$$

$$\lambda^{n}_{13}=\lambda^{n}_{\text{проф}}, \qquad (4.17)$$

$$\lambda^{\text{л}}_{31} = \frac{a(1-p^{\text{л}})\left[1-e^{-(X_5^{\text{г}}+X_3^{\text{л}})}\right]}{X_5^{\text{г}}+X_3^{\text{л}}}, \tag{4.18}$$

$$\lambda^{\text{л}}_{34} = \frac{ap^{\text{л}}\left[1-e^{-(X_5^{\text{г}}+X_3^{\text{л}})}\right]}{X_5^{\text{г}}+X_3^{\text{л}}}, \tag{4.19}$$

$$\lambda^{\text{л}}_{54} = \nu^{\text{л}}, \tag{4.20}$$

$$\lambda^{\text{л}}_{41} = \frac{b\left[1-e^{-(X_6^{\text{г}}+X_4^{\text{л}})}\right]}{X_6^{\text{г}}+X_4^{\text{л}}}. \tag{4.21}$$

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Thus, all the intensities of flow of events for both sub-graphs of Fig. 6.16 are found.

Substituting in them for the number of states by average numbers, let us write the system of the differential equations of the dynamics of average in the form:

$$\left.\begin{aligned}
\frac{dm_1^{\text{г}}}{dt} &= -\lambda^{\text{г}} R(m_1^{\text{г}}) - \lambda^{\text{г}}_{\text{проф}} m_1^{\text{г}} + \frac{1}{\bar t^{\text{г}}_{\text{пор}}} m_4^{\text{г}} + \\
&+ \frac{a(1-p^{\text{г}})\left[1-e^{(m_5^{\text{г}}+m_3^{\text{л}})}\right] m_5^{\text{г}}}{m_5^{\text{г}}+m_3^{\text{л}}} + \frac{b\left[1-e^{-(m_6^{\text{г}}+m_4^{\text{л}})}\right] m_6^{\text{г}}}{m_6^{\text{г}}+m_4^{\text{л}}}; \\
\frac{dm_2^{\text{г}}}{dt} &= -\left(\nu^{\text{г}}_{\text{пор}} + \frac{1}{\bar t^{\text{г}}_{\text{пор}}}\right) m_2^{\text{г}} + \lambda^{\text{г}} R(m_1^{\text{г}}); \\
\frac{dm_3^{\text{г}}}{dt} &= -\left(\nu^{\text{г}}_{\text{гр}} + \frac{1}{\bar t^{\text{г}}_{\text{гр}}}\right) m_3^{\text{г}} + \frac{1}{\bar t^{\text{г}}_{\text{пор}}} m_2^{\text{г}}; \\
\frac{dm_4^{\text{г}}}{dt} &= -\left(\nu^{\text{г}}_{\text{пор}} + \frac{1}{\bar t^{\text{г}}_{\text{пор}}}\right) m_4^{\text{г}} + \frac{1}{\bar t^{\text{г}}_{\text{гр}}} m_3^{\text{г}}; \\
\frac{dm_5^{\text{г}}}{dt} &= -\frac{a\left[1-e^{-(m_5^{\text{г}}+m_3^{\text{л}})}\right] m_5^{\text{г}}}{m_5^{\text{г}}+m_3^{\text{л}}} + \lambda^{\text{г}}_{\text{проф}} m_1^{\text{г}}; \\
\frac{dm_6^{\text{г}}}{dt} &= -\frac{b\left[1-e^{-(m_6^{\text{г}}+m_4^{\text{л}})}\right] m_6^{\text{г}}}{m_6^{\text{г}}+m_4^{\text{л}}} + \\
&+ \frac{a p^{\text{г}}\left[1-e^{-(m_5^{\text{г}}+m_3^{\text{л}})}\right] m_5^{\text{г}}}{m_5^{\text{г}}+m_3^{\text{л}}} + \nu^{\text{г}}_{\text{пор}}(m_2^{\text{г}}+m_4^{\text{г}}) + \nu^{\text{г}}_{\text{гр}} m_3^{\text{г}}; \\
\frac{dm_1^{\text{л}}}{dt} &= -\lambda^{\text{л}} R(m_1^{\text{л}}) - \lambda^{\text{л}}_{\text{проф}} m_1^{\text{л}} + \frac{1}{\bar t^{\text{л}}} m_2^{\text{л}} + \\
&+ \frac{a(1-p^{\text{л}})\left[1-e^{-(m_5^{\text{г}}+m_3^{\text{г}})}\right] m_3^{\text{л}}}{m_5^{\text{г}}+m_3^{\text{л}}} + \frac{b\left[1-e^{-(m_6^{\text{г}}+m_4^{\text{л}})}\right] m_4^{\text{л}}}{m_6^{\text{г}}+m_4^{\text{л}}}; \\
\frac{dm_2^{\text{л}}}{dt} &= -\left(\frac{1}{\bar t^{\text{л}}} + \nu^{\text{л}}\right) m_2^{\text{л}} + \lambda^{\text{л}} R(m_1^{\text{л}}); \\
\frac{dm_3^{\text{л}}}{dt} &= -\frac{a\left[1-e^{-(m_5^{\text{г}}+m_3^{\text{л}})}\right] m_3^{\text{л}}}{m_5^{\text{г}}+m_3^{\text{л}}} + \lambda^{\text{л}}_{\text{проф}} m_1^{\text{л}}; \\
\frac{dm_4^{\text{л}}}{dt} &= -\frac{b\left[1-e^{-(m_6^{\text{г}}+m_4^{\text{л}})}\right] m_4^{\text{л}}}{m_6^{\text{г}}+m_4^{\text{л}}} + \\
&+ \nu^{\text{л}} m_2 + \frac{a p^{\text{л}}\left[1-e^{-(m_5^{\text{г}}+m_3^{\text{л}})}\right] m_3^{\text{л}}}{m_5^{\text{г}}+m_3^{\text{л}}}.
\end{aligned}\right\} \quad (4.22)$$

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Recall that in these equations R(x) - the function, determined by formula (4.4).

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Thus, you wrote ten differential equations for the average numbers of states (6 + 4 = 10). However, actually to solve is necessary not so much: any of six first equations and any of four latter can be reject/thrown corresponding function it is expressed from the condition:

$$m_1^{\Gamma}+m_2^{\Gamma}+m_3^{\Gamma}+m_4^{\Gamma}+m_5^{\Gamma}+m_6^{\Gamma}=N^{\Gamma}, \quad m_1^{Л}+m_2^{Л}+m_3^{Л}+m_4^{Л}=N^{Л}.$$

Thus, the total number of differential equations, which it is necessary to solve, is equal to eight (5 + 3 = 8).

The initial conditions under which we will solve this system, depend on which question we wish to explain. If, for example, us interests chiefly the initial operating cycle of basis, soon after its organization, it is logical to assume that at the initial moment all machines are located in states Γ_1 and $Л_1$; then initial conditions will be:

$$t=0; \quad m_1^{\Gamma}=N^{\Gamma}; \quad m_2^{\Gamma}=m_3^{\Gamma}=m_4^{\Gamma}=m_5^{\Gamma}=m_6^{\Gamma}=0;$$
$$m_1^{Л}=N^{Л}; \quad m_2^{Л}=m_3^{Л}=m_4^{Л}=0.$$

But if us interests another, for example, how rapidly system can manage the "block", caused by the large number of malfunctions, it is possible to assume that at the initial moment the already large number of machines is located in repair (state Γ_6 and $Л_4$).

Let us focus attention on the fact that the obtained by us system of differential equations for the average numbers of states is

nonlinear. This is very typical for the method of the dynamics of average under conditions when the intensities of flow of events depend on the numbers of states. Nevertheless the solution of this system differential equations by ETsVM or even by hand (it is numerical) difficulties does not represent. For this, it is only necessary to assign the numerical values of all parameters, which figure in problem.

5. The asymptotic behavior of the average numbers of states.

In the previous paragraphs we considered the methodology of the description of the process, taking place in the complex (multiunit) system S with the help of the equations of the dynamics of the average, on which unknown functions are the average numbers of states: $m_1, m_2, \ldots, m_n$.

It is logical, does arise the question: to which limiting values they do strive (if do strive) these average numbers with $t \to \infty$? There does exist, and if there does exist, then which steady-state conditions/mode?

In the case when we examined equations for the probabilities of states, a question on maximum conditions/mode was solved sufficiently simply.

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If from any state system could pass into any another, and the number of states was certain, then there was maximum steady-state conditions/mode, not depending on initial conditions. In order to find the probabilities of states in this conditions/mode, it suffices it was to place the left sides of the differential equations equal to zero and to solve the obtained system of linear algebraic equations.

For the method of the dynamics of average, the matter is somewhat more complex. Recall that the equations of the dynamics of average in the general case are nonlinear; so are nonlinear the algebraic equations, obtained of them, if left sides are assumed equal to zero. It can seem that the solution of this system of equations not is singular, then it is necessary to consider the set of solutions and to reject/threw those of them, that do not answer the physical conditions of problem. Even if solution singularly, it is necessary all the same to trace the behavior of the solution of the system of differential equations with $t \to \infty$.

Let us demonstrate the special feature/peculiarity of research of the asymptotic behavior of the average numbers of states based on

the example of the problem, undertaken, for diversity, from the field of biology. Let in certain locality dwell animals of two forms A and B, moreover animals of the first form (A) - predatory, and they are fed by animals of the second form (B), which are satisfied by vegetable food.

Let us characterize the state of each animal maximally roughly, taking into account only, lively it still or it perished. The construction of the graph/count of the states of the elements of system is caused no difficulties: this graph/count will be decomposed into two subgraph, that correspond to forms A and B (Fig. 6.19). Here the arrow/pointers, which lead from A_1 in A_2 and from B_1 and B_2, consider the mortality of animals, moreover for form B - mortality of two kinds: and those cases when individual perishes by natural death, and those, when it eats up animal of form A. Double half-arrows, which lead into states A_1, B_1, correspond to the addition/completion of numbers because of birth rate.

Let us designate the numbers of cell/elements (animals) in states A_1, A_2, B_1, B_2 respectively through $X_1^A, X_2^A, X_1^B, X_2^B$, and their mathematical expectations through $m_1^A, m_2^A, m_1^B, m_2^B$. We wish to comprise differential equations for m_1^A, m_1^B (average numbers m_2^A, m_2^B the previous generations as finally leaving from "active" cell/elements systems, will not enter in these equations, and we can by them be

interested in no way).

In order to comprise differential equations, it is necessary to assign the form of the dependence of intensities of flow λ_{12}^{A}, λ_{12}^{B}, δ^{A}, δ^{B} on X_1^{A}, X_1^{B}.

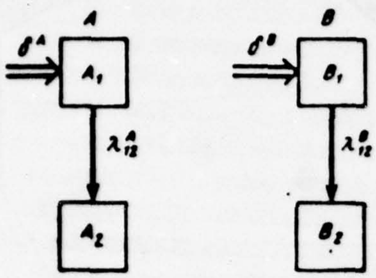

Fig. 6.19.

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Let us begin from herbivorous animals of form B. Let us assume that the supplies of the food, available to them, depend on the number of animals not of form A, of form B, and which the food suffices on all. Then it is logical to assume that the average birth rate per unit time (in recalculation to one living individual of form B) remains constant. Let us designate this the constant c; then growth of the number of state B_1 because of the generation of new animals it will be expressed by the formula:.

$$\delta^B = cX_1^B. \tag{5.1}$$

As concerns mortality λ_{12}^B, then we already said that it is composed of two terms. The first - this is natural mortality; we will consider it constant (on the same reasons, on which they considered constant birth rate). Let us designate this constant (average number of natural deaths in recalculation to one living individual of form

B) through k; value k can be interpreted as portion/fraction of animals of form B, which perish per unit time by natural death. Second term in composition λ_{12}^{B} - this portion/fraction of animals of form B, eaten up per unit time by the plunderers of form A. It is logical to assume that the number of meetings (per unit time) of the animal forms A and B, which end with the fact that A eats up B, directly proportional to the number X_1^A of animals of form A (living) and to the number X_1^B of the available animals of form B, i.e., that this number is expressed by the formula

$$lX_1^A \cdot X_1^B, \tag{5.2}$$

where l — a constant. In recalculation for one living individual of form A the number of such deaths "devourings") per unit time will be equal to lX_1^A, so that the intensity of flow of deaths, which is necessary to one individual of form B in state B_1, will be expressed as follows:.

$$\lambda_{12}^{B} = k + lX_1^A. \tag{5.3}$$

We will be now occupied by animals of form A. We assumed that their only food comprises form B; therefore it is logical that both the birth rate and the mortality of form A they will depend on the number of those eaten up per unit time of animals, that are necessary to one predator. The number of eaten up animals is assigned by formula (5.2), and to their one plunderer it comes on the average lX_1^B. Thus, the mortality of plunderers λ_{12}^{A} will be some function from lX_1^B, or, since l — a constant, by some function from X_1^B:

$$\lambda_{12}^{A} = f(X_1^B). \tag{5.4}$$

It is obvious, function f will be decreasing function from X_1^B (i.e. from a quantity of food). It is real/actual, animals will that lesser die, than more they will have food. The possible form of the function f is represented in Fig. to 6.20.

It is analogous, birth rate will be some other function from the number of animals of form $B: g(X_1^B)$, and the average increase of the population of plunderers because of birth rate (per unit time) will be registered as follows:.

$$\delta^A = X_1^A g(X_1^B). \qquad (5.5)$$

Function g, it is logical, increasing function from X_1^B (Fig. 6.21).

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Let us write out now differential equations for the average numbers of states:

$$\left.\begin{aligned} \frac{dm_1^A}{dt} &= -\lambda_{12}^A m_1^A + \delta^A, \\ \frac{dm_1^B}{dt} &= -\lambda_{12}^B m_1^B + \delta^B. \end{aligned}\right|$$

Using the principle of quasi-regularity and substituting for number X_1^A, X_1^B on which they depend intensity (5.1), (5.3), (5.4),

(5.5), by their average values m_1^A, m_1^B, we will obtain:

$$\left.\begin{aligned} \frac{dm_1^A}{dt} &= -f(m_1^B)\, m_1^A + g(m_1^B)\, m_1^A, \\ \frac{dm_1^B}{dt} &= -(k + l m_1^A)\, m_1^B + c m_1^B. \end{aligned}\right\} \quad (5.6)$$

We see that in the first equaticn actually enters only a difference in functions g and of f, which characterizes full/total/complete average increase. Let us designate this difference through h:

$$h(x) = g(x) - f(x). \quad (5.7)$$

This a difference in increasing and decreasing function, and that means that function itself h - increasing (see Fig. 6.22). Unlike the positive functions f and g, this function can reverse the sign. Let us assume that it reverses the sign at some point x_0^B (Fig. 6.22). Value x_0^B makes sense of that number of animals of form B, with which the birth rate and the mortality of plunderers on the average are balanced. At the high values of the number of form B, the number of form A on the average grows, with smaller - on the average it decreases. Let us name x_0^B the critical number of form B.

In exactly the same way it is possible to introduce the critical number x_0^A of plunderers, so on of which the number of herbivorous animals on the average remains constant/invariable.

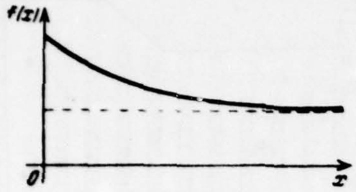

Fig. 6.20.

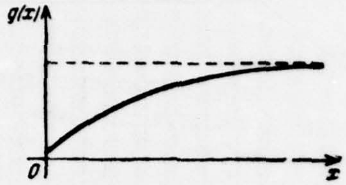

Fig. 6.21

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From second equation (5.6) it is evident that

$$x_0^A = \frac{c-k}{l}. \tag{5.8}$$

When the number of plunderers is greater than x_0^A, then the number of form B decreases, but with their smaller number - grows.

Using new designations, the equations of the dynamics of average (5.6) can be rewritten in the form:

$$\left.\begin{aligned} \frac{dm_1^A}{dt} &= h(m_1^B)\, m_1^A; \\ \frac{dm_1^B}{dt} &= l\,(x_0^A - m_1^A)\, m_1^B. \end{aligned}\right\} \tag{5.9}$$

In this form we will analyze equations for the average numbers of states of system.

Let us look how with change t will move the point on plane, which represents solution m_1^A, m_1^B system (5.9) (see Fig. 6.23).

When $m_1^B < x_0^B$, i.e., it is lower than the straight line $m_1^B = x_0^B$, according to first equation (5.9), $\frac{dm_1^A}{dt} < 0$; this indicates that point moves to the left (it is simultaneous with this perhaps upward or downward). Higher than this direct/straight point moves to the right. In exactly the same manner from second equation (5.9) we obtain, that is more left than the straight line $m_1^A = x_0^A$ the motion of point directed upward, and more to the right by this straight line - downward. In the torque/moments of the passage of straight line $m_1^B = x_0^B$ point moves strictly vertically: downward, if it is located more to the right x_0^A and upward - if more left; at the moment of the passage of straight line $m_1^A = x_0^A$ it moves strictly horizontally.

Thus, the point, which represents the solution of system (5.9), revolves around point (x_0^A, x_0^B) clockwise, moreover it revolves, generally speaking, not in circle - it can be, approaching a point (x_0^A, x_0^B), but it can be receding from it. Point (x_0^A, x_0^B) - position of equilibrium: if we as initial conditions for the solution of system (5.9) take x_0^A, x_0^B, then number m_1^A, m_1^B they will not be changed, and the solution of system (5.9) does not depend on the time:

$$m_1^A(t) \equiv x_0^A; \quad m_1^B(t) \equiv x_0^B.$$

This position of equilibrium can be stable, if the solutions of system, which begin from the points, close to (x_0^A, x_0^B), in the course

of time unlimitedly approach this point; and unstable - if they recede.

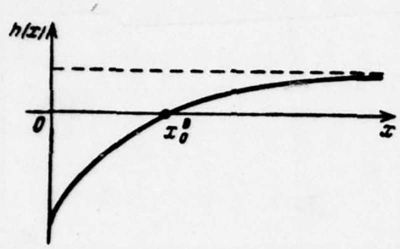

Fig. 6.22.

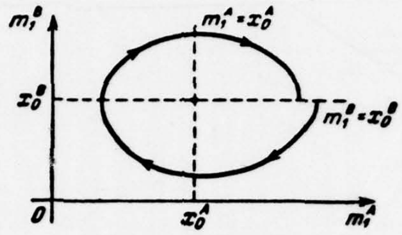

Fig. 6.23

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Figures 6.23 depicts the case when the solution, which begins on straight line $m_1{}^B = x_0{}^B$ through one "revolution" proves to be nearer to point $(x_0{}^A, x_0{}^B)$, than in the beginning. It is obvious, are possible an additional two cases:

- after one "revolution" point will render/show further from $(x_0{}^A, x_0{}^B)$, than in the beginning;

- after one "revolution" point in accuracy will return to its initial position.

In the latter case all the subsequent "revolutions" of point $m_1{}^A, m_1{}^B$ around position of equilibrium they will be accomplished on the one and the same locked way. Such the locked way in the theory of

differential equations is called cycle; to it corresponds periodic solution $m_1^A(t), m_1^B(t)$: the number of plunderers grows; therefore the number of herbivorous begins to decrease, it reaches x_0^B; then begins to decrease the number of plunderers, and, when it passes critical value x_0^A, the number of herbivorous begins to grow; finally, the numbers of each animals reach their previous values and process begins anew. Just as position of equilibrium, cycle can be stable (when the solutions, which begin near blisses, unlimitedly approach a cycle) and unstable (when they recede).

Let us return to the case, depicted in Fig. 6.23. The following revolution of point (m_1^A, m_1^B) around position of equilibrium is still more it approaches to point (x_0^A, x_0^B), and with each following revolution point (m_1^A, m_1^B) will all more approach a point (x_0^A, x_0^B). Here furthermore there can be two cases: or point will unlimitedly approach a position of equilibrium - then obviously, this position of equilibrium will be stable; or it will with each revolution approach, but not it is unlimited. In this case point (m_1^A, m_1^B) will unlimitedly approach outside some cycle which, of course, will be stable.

The same reasonings can be used, also, in the case when after one revolution point (m_1^A, m_1^B) recedes from position of equilibrium: or in the course of time it will unlimitedly recede from point (x_0^A, x_0^B), and also, therefore, from the origin of coordinates; or it will

approach some stable cycle (this time from within).

All these characters of the behavior of the solutions of system (5.9) can differently be combined with each other, forming sometimes sufficiently complex picture. For example, is represented on Fig. 6.24 case when position of equilibrium is stable, and there are no cycles (periodic solutions). There will here be only maximum conditions/mode (x_0^A, x_0^B); and to it will strive all the solutions, whateverwere initial conditions. Figures 6.25 depicts another case when position of equilibrium is unstable, and cycles, as in the preceding case, no. In this case of no maximum conditions/mode exists: the number of each form first decreases almost to zero, then strongly it grows, moreover with each "revolution" these vibrations become all more, after growing unlimitedly. It goes without saying that in accuracy this character of the asymptotic behavior of the numbers of forms virtually cannot be met.

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Here will begin to manifest itself the not taken into consideration during formalization problems the factors: for example, the limitedness of the supplies of food of the herbivorous or error, connected with the application/use of principle of the quasi-regularity (which will be considerable at low values m_1^A, m_1^B).

One Additional Example of the possible character of the asymptotic behavior of the solutions: position of equilibrium is unstable, there is one stable cycle C (see Fig. 6.26). There will here be again only maximum conditions/mode - periodic increase and decrease of numbers, which corresponds to limiting cycle C. To this conditions/mode approach average numbers $m_1^A(t), m_1^B(t)$ under any initial conditions.

Figures 6.27 depicts the even more complex case: stable position of equilibrium, around it the unstable cycle C_1, and around it - one additional, but the stable cycle C_2. Maximum conditions/modes here two: position of equilibrium (x_0^A, x_0^B) and the periodic behavior, which corresponds to cycle C_2. To the first maximum conditions/mode of number m_1^A, m_1^B approach, if initial conditions are located within cycle C_1 (shaded region in Fig. 6.27); to the second - if at first point (m_1^A, m_1^B) it was located out of cycle C_1.

It is certain, it can seem that some of the enumerated cases, during the specific limitations to the form of the function h(x) (for example, for bounded function h(x)) are impossible. But, in any case, the picture of the asymptotic behavior of solutions will be not always one and the same, and for the determination of maximum

conditions/modes is necessary the study of system (5.9) with the concrete/specific/actual values of the entering it numerical parameters and the concrete/specific/actual form of the function h(x).

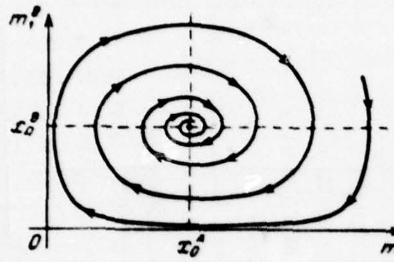

Fig. 6.24.

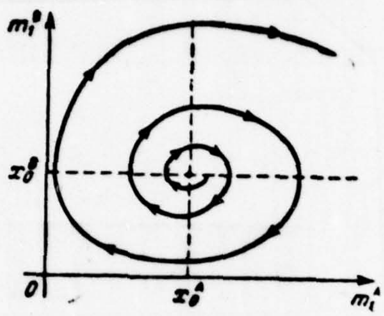

Fig. 6.25.

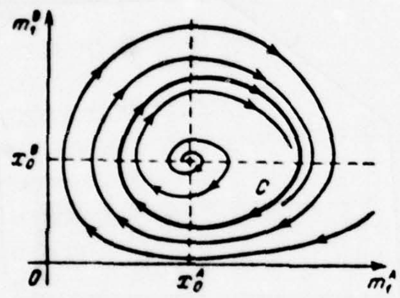

Fig. 6.26.

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Example. Under conditions of the examined above problem function **h(x)**, assigning is the relative increase of the number of plunderers per unit time depending on the number of herbivorous, has the form:

$$h(x) = 1 - \frac{3}{2000 + x};$$

constants l and x_0^A are equal to with respect 0.001 and 1000. In this case, the equations of the dynamics of average (5.9) take the following form:

$$\left.\begin{aligned} \frac{dm_1^A}{dt} &= \left[1 - \frac{3}{2000 + m_1^B}\right] m_1^A, \\ \frac{dm_1^B}{dt} &= \left[1 - \frac{m_1^A}{1000}\right] m_1^B. \end{aligned}\right\} \qquad (5.10)$$

Let us note that x_0^B (the "critical" number of animals of form B, with which animals of form A in average/mean not multiply and they do not die out), in this case also equal to 1000.

Solving numerically system (5.10) under different initial conditions, we are convinced that here occurs the case, depicted in Fig. 6.26: there is an unstable position of equilibrium (1000, 1000) and one cycle, moreover stable. This cycle will play the role of maximum conditions/mode, for any initial conditions. As show calculations, the period of this maximum it is solved it will be approximately equal to 10.8 time units; a change of the average numbers of forms A and B during this time is represented in Table 5.1, and graphically - in Fig. 6.28 and 6.29. As the zero time reference, is undertaken the torque/moment of the smallest number of herbivorous animals.

Figures 6.28 depicts the dependence of the average numbers of plunderers (m_1^A) and herbivorcus (m_1^B) on time t; Fig. 6.29 shows stable limiting cycle. Cycle is labeled on time; divisions correspond t = 1, 2, ..., 10. With t = 10.8 (period of cycle) m_1^A and m_1^B they are returned to initial values.

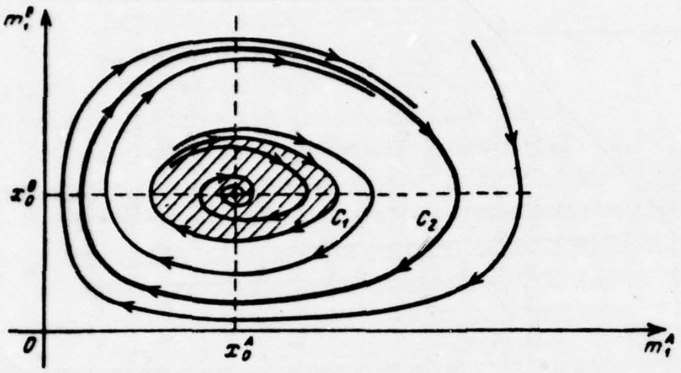

Fig. 6.27.

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Analyzing data given in Table 5.1, we see that at first, when the number of herbivorous is small (into average/mean 450), plunderers starve, their number decreases and therefore the number of herbivorous begins to increase. At that torque/moment when the number of herbivorous reaches 1000 (this occurs somewhat more than through 3 time units from the beginning of cycle), the average number of plunderers it reaches its minimum (about 620). After herbivorous it becomes more than 1000, the number of plunderers again begins to increase and again reach 1000 somewhat later than t = 6. The number of herbivorous at this time it reaches the maximum (2170 or somewhat more), and then again it begins to decrease, because the number of plunderers is more critical 1000. Then (with t between 8 and 9) the

number of plunderers reaches the maximum, equal to approximately 1550, and the number of herbivcrous at this time is equal to critical 1000.

Table 5.1.

(1) Время t	0	1	2	3	4	5	6	7	8	9	10	10,8
(2) Средняя численность хищников m_1^A	1000	790	670	620	650	750	960	1270	1540	1490	1210	1000
(3) Средняя численность травоядных m_1^B	450	500	650	930	1310	1810	2170	1950	1200	670	480	450

Key: (1). Time t. (2). Average number of plunderers. (3). Average number of herbivorous.

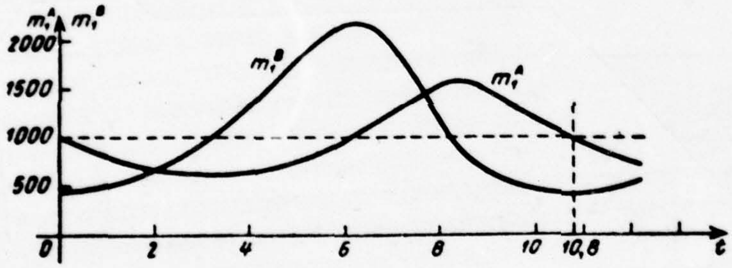

Fig. 6.28.

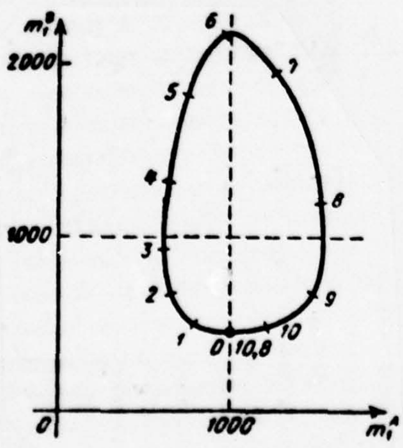

Fig. 6.29

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On the last/latter section of cycle (tc torque/moment t = 10.8) decreases the number of herbivorous (because the number of plunderers exceeds 1000), and the number of plunderers (because herbivorous less than 1000). At the end of the period, with t = 10.8, the number of plunderers again reaches critical value 1000, and the number of herbivorous is returned to its smallest value with 450.

It is interesting that in nature real/actually are encountered such alternations in the numbers in connected with each other forms.

6. Equations of combat dynamics (model A).

The method of the dynamics of average can be successfully used for the approximate description of the processes of the combat operations in which participate the numerous groups of one or the other cell/elements (tanks, ships, aircraft, etc.). Moreover, precisely the description of the processes of combat operations ("combat dynamics") - one of the first on time application/uses of a method of the dynamics of average. The differential equations, which describe a change of the numbers of fighting groups in the process of

breakage, by the name of the "equations of Lanchester", appeared even in the times of the first world war. True, the field of their application/use was then is very narrow (a total of two-three of model), but communication/connection of method with the Markovian processes not established. At present the method of the dynamics of average received wide development and represents by itself the detailed and very flexible vehicle, which makes it possible to describe the most diverse combat situations (see for example [1, 11, 13, 23]).

Here we will consider only a little from the tasks of combat dynamics, predominantly at systematic visual angle, without dwelling in detail on the quantitative side of dependences.

We will examine the combat situations, in which clash the groupings, which consist of a large quantity of cell/elements which we will call "combat units" (aircraft, tanks, ships, rocket installations, etc.) besides combat units, in some models will participate "auxiliary unit" (radar stations, prospectors, confusing reflectors, etc.) whose difference from the combat units - in the fact that they not can news themselves fire/light on the objects of the enemy, accomplishing different ensuring tasks.

Systems mathematical model, we will examine the described

phenomena within the framework of Markov random chains (with the escape/ensuing of them method of the dynamics of average). Therefore we always will assume that each combat unit produces the Poisson flow of shots with certain intensity λ, which can be both the constant and variable, depending on time. During the calculation of this intensity, it is necessary to take into attention not simply the "rate of fire" of combat unit, but its actual average rate of fire, taking into account the time, required for the calculation of sighting data, aiming, recharging it is other.

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If the shooting of combat unit is conducted on the uniform target/purposes each of which as a result of shot on it can be only "struck" or "struck" ("damage/defeat" indicates breakdown), then it is convenient instead of the rate of fire λ to use the efficient rate of fire

$$\lambda^* = \lambda p,$$

where p - kill probability of unit by the directed along it shot. Value λ represents by itself nothing else but the intensity of flow "successful" (damaging) of shots, produced by one combat unit.

Calculations show that in the examination of the dynamics of combat of numerous groups the assumption about the Poisson character

of the flow of shots (or successful shots) does not distort any seriously the picture of phenomenon. Furthermore, one must take into account that the task of the method of the dynamics of average - the creation not of the detailed and precise, but roughly approximate model of breakage.

Let us examine the first following simplest model of breakage - let us name it "model A". In breakage take part two groupings: K (are redder) and C (blue) (Fig. 6.30). Let us note the parameters, which relate to redder and blue by the superscripts of "K" and «C». In the composition of grouping K is N^{K} uniform combat units (aircraft, tanks, ships), in the composition of grouping C — N^{C} the combat units, uniform between themselves, but are not compulsory uniform with combat units redder. The efficient rate of fire of one combat unit redder is equal to λ^{K}, blue - λ^{C}.

Relative to the organization of breakage we take following assumptions.

1. Each combat unit redder can conduct fire/light on each combat unit of blue, and vice versa.

2. Fire/light is sighting, i.e., it is directed on completely specific combat unit; by one shot cannot be struck more than one unit.

3. Bombardment undergoes with identical probability any of still not affected units; after damage of unit, fire/light on it ceases and immediately it is transferred to other, still that not affected.

4. Affected unit ceases shooting and in further combat operations does not participate.

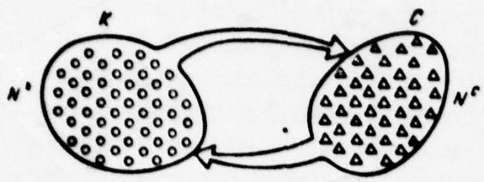

Fig. 6.30.

Fig. 6.31

Fig. 6.31.
Key: (1). It is not struck. (2). It is struck.

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Thus, in our simplest model each combat unit can be in one of the two states: "it is not struck" (and, which means, that conducts fire/light) and "struck" (it ceased fire). The graph/count of the states of the elements of system, divided into two subgraph K and C, is shown on Fig. 6.31. Letters $\lambda_{12}^к$, λ_{12}^c designated the intensities of flow of the events, which translate cell/element (combat unit) from state into state.

Let us designate, as ever,

$$X_1^к = X_1^к(t); \quad X_2^к = X_2^к(t);$$
$$X_1^c = X_1^c(t); \quad X_2^c = X_2^c(t)$$

the numbers of states K_1, K_2, C_1, C_2 at the moment of time t. Through

m_1^{κ}, m_2^{κ}, m_1^{c}, m_2^{c} we will designate the appropriate average numbers.

It is obvious, in the case of intensity λ_{12}^{κ}, λ_{12}^{c} in question vary and depend in the course of time on the numbers of states (quantity of shooting units). Let us determine these intensities. Let us discuss as follows. Each combat unit of blue produces per unit time λ^c of successful shots. In torque/moment t shoots X_1^c the combat units of blue; everything together per unit time they give on the average

$$\lambda^c X_1^c$$

of successful shots. These shots are distributed evenly between all preserved up to given torque/moment combat units redder, so that for each of them it comes on the average

$$\frac{\lambda^c X_1^c}{X_1^{\kappa}} \qquad (6.1)$$

of successful shots. But this still not all: intensity (6.1) must be multiplied by function $R(X_1^{\kappa})$ (see formula (4.4) §4), which it is converted into zero with $X_1^{\kappa} = 0$ (if at torque/moment t at redder was preserved not one combat unit, blue simply not on coma it will shoot).

Taking into account that $R(x)/x = \rho(x)$ (see formula (4.5) §4), we will obtain

$$\lambda_{12}^{\kappa} = \lambda^c X_1 \rho(X_1^{\kappa}). \qquad (6.2)$$

Analogously we find

$$\lambda_{12}^{c} = \lambda^{\kappa} X_1^{\kappa} \rho(X_1^{c}). \qquad (6.3)$$

Knowing these intensities and using the principle of quasi-regularity, it is possible on graph/count's basis (see Fig. 6.31) to immediately write the equations of the dynamics of average:

$$\left.\begin{aligned}\frac{dm_1^{\kappa}}{dt} &= -\lambda^{c} m_1^{c} R(m_1^{\kappa}),\\ \frac{dm_1^{c}}{dt} &= -\lambda^{\kappa} m_1^{\kappa} R(m_1^{c}).\end{aligned}\right\} \qquad (6.4)$$

Equations for m_2^{κ}, m_2^{c} we do not write, since for any t

$$m_1^{\kappa} + m_2^{\kappa} = N^{\kappa}; \quad m_1^{c} + m_2^{c} = N^{c}.$$

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Let us note that, as a rule, us do not interest the number destroyed units m_2^{κ}, m_2^{c}, since active participation in combat operations they do not accept.

To solve equations (6.4) is possible under any initial conditions; they usually assume that at the initial moment all units are whole:

$$t=0; \quad m_1^{\kappa}=N^{\kappa}, \quad m_1^{c}=N^{c}.$$

Let us focus attention on the fact that in the initial stages of breakage, distant from the stage of "exhaustion", the average number of cell/elements in states K_1, C_1 is more one, the value of functions $R(m_1^{\kappa})=R(m_1^{c})=1$, and instead of equations (6.4) it is possible to register:

$$\left.\begin{aligned}\frac{dm_1^{\kappa}}{dt}&=-\lambda^{c}m_1^{c},\\ \frac{dm_1^{c}}{dt}&=-\lambda^{\kappa}m_1^{\kappa}.\end{aligned}\right\}\quad(6.5)$$

Equations (6.5) are known in the literature as of equation of Lanchester of 2nd kind. It should be noted that such equations, even in more precise form (6.4), are suitable for describing combat

dynamics only at its initial stages when the average numbers of both groupings are not still small in comparison with their initial numbers, but in far progressed stages of breakage (stage of "exhaustion") they cease to be suitable even approximately[1].

FOOTNOTE [1]. On the evaluation of the errors, connected with the principle of quasi-regularity, see further, § 13. ENDFOOTNOTE.

Let us note that, unlike equations (6.4), equations (6.5) are linear, which represents essential advantage during their solution.

When deriving the equations (6.4), (6.5) we in any way did not specify, were constant or alternating/variable efficient rates of fire $\lambda^к, \lambda^с$ - equations are valid both in that and in other case. However, with constant efficient rates of fire $(\lambda^к = \text{const}, \lambda^с = \text{const})$ of equation (6.5) it is possible to integrate in an explicit form. Lowering elementary transformations, let us give directly the final result:

$$\left.\begin{aligned} m_1^к &= N^к \operatorname{ch}\sqrt{\lambda^к\lambda^с}\,t - N^с\sqrt{\frac{\lambda^с}{\lambda^к}}\operatorname{sh}\sqrt{\lambda^к\lambda^с}\,t, \\ m_1^с &= N^с \operatorname{ch}\sqrt{\lambda^к\lambda^с}\,t - N^к\sqrt{\frac{\lambda^к}{\lambda^с}}\operatorname{sh}\sqrt{\lambda^к\lambda^с}\,t, \end{aligned}\right\} \quad (6.6)$$

where $\operatorname{ch} x = \frac{1}{2}(e^x + e^{-x}),\ \operatorname{sh} x = \frac{1}{2}(e^x - e^{-x})$ - hyperbolic functions.

Curves $m_1^к(t), m_1^с(t)$ have a different type depending on the initial relationship/ratio of forces $N^к/N^с$ and relationship/ratios of efficient rates of fire $\lambda^к/\lambda^с$.

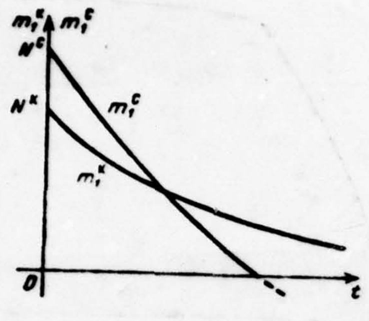

Fig. 6.32.

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For example, Fig. 6.32 shows the case when in the beginning of breakage blue have quantitative advantage above redder ($N^c > N^к$), a in the course of breakage red[redacted] they conquer, because of larger efficient rate of fire ($\lambda^к > \lambda^c$). Let us focus attention on the fact that the curve $m_1^c(t)$ (number of conquered side) approaches the axis of abscissas at angle and during continuation it would cross it, i.e., the average number of conquered side would become negative, which, it goes without saying, is impossible. This occurs because for the final stages of the breakage when side C is close to the state of exhaustion, equation (6.5), as we already spoke, they cease to be those used. If we solved not equations (6.5), but more precise equations (6.4), curve $m_1^c(t)$ smoothly it would approach an axis 0t.

Analyzing the solution of Lanchester's equations (6.6), it is

possible tc trace, as they affect this solution of the condition of breakage (parameters $N^к$, N^c, $\lambda^к$ and λ^c). For this let us divide equations (6.6) on N^K and N^C let us pass to relative quantities of preserved combat units at torque/moment t:

$$\left.\begin{aligned} \mu_1{}^к &= \mathrm{ch}\sqrt{\lambda^к\lambda^c}\,t-\frac{N^c}{N^к}\sqrt{\frac{\lambda^c}{\lambda^к}}\,\mathrm{sh}\sqrt{\lambda^к\lambda^c}\,t,\\ \mu_1{}^c &= \mathrm{ch}\sqrt{\lambda^к\lambda^c}\,t-\frac{N^к}{N^c}\sqrt{\frac{\lambda^к}{\lambda^c}}\,\mathrm{sh}\sqrt{\lambda^к\lambda^c}\,t.\end{aligned}\right\} \quad (6.7)$$

From formulas (6.7) it is evident that the decrease of the numbers of each of the sides in larger measure depends on the relationship/ratio of forces N^C/N^K than from the relationship/ratio of efficient rates of fire $\lambda^c/\lambda^к$ (first sense enters in formulas (6.7) directly, and the second - under radical sign). This completely explainable: it is real/actual, during that organization of breakage who is accepted in our model A (shooting is conducted only on nonafflicted unity) red[redacted] more advantageous, for example, to twice increase the number of unity N^K than to twice increase the efficient rate of fire of each $\lambda^к$: to the damage of two unity the enemy is forced to spend double more resources, than to damage/defeat to one.

The more detailed analysis of the solution of the equations of Lanchester of the 2nd kind does not enter in our tasks; the reader interesting can be referred to to management/manuals [11, 13, 23].

7. Account of the addition/completion of forces, preventive attack, rate of the mobilization and of other factors at the equations of combat dynamics.

In the equations of combat dynamics, it is possible very to simply take into account different factors, which relate to the organization of combat operations as that:

- input/introduction of reserves (addition/completion of forces),

- preventive attack by one of the sides;

- rate of the mobilization of combat devices,

- an exhaustion of the ammunition

and of so forth.

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Let us show how this it is possible to do. Let in the process of breakage each of the sides introduce into action reserves in a quantity $\delta^{\text{к}}$ of combat units per unit time (for red[redacted]) and $\delta^{\text{с}}$ - for blue. We already be able (see §3) to consider in the equations of the dynamics of average the addition/completion of the composition of

cell/elements from without. into datum the input/introduction of reserves will be taken into account with the help of the additional component in the right side of each equation of dynamics breakage:

$$\left.\begin{aligned}\frac{dm_1^{\kappa}}{dt} &= -\lambda^{c} m_1^{c} + \delta^{\kappa},\\ \frac{dm_1^{c}}{dt} &= -\lambda^{\kappa} m_1^{\kappa} + \delta^{c}.\end{aligned}\right\} \quad (7.1)$$

Values $\delta^{\kappa}, \delta^{c}$ can be both the constants and variables both depending and not depending on the average numbers of sides. It is solved these equations and analyzing the course of changing the numbers of sides (Fig. 6.33), it is possible to do conclusion about the rational rate of the input/introduction of reserves, the period of its beginning and termination.

In equations of combat dynamics, it is possible to take into account not only addition/completion of forces, but also a series of other factors: preventive attack, the rate of mobilization, the exhaustion of cash resources of ammunition and its restoration, etc. For entire this are sufficient to set/assume in the equations of combat dynamics efficient rates of fire $\lambda^{\kappa}, \lambda^{c}$ constants, but varying for the specific law:

$$\lambda^{\kappa} = \lambda^{\kappa}(t); \quad \lambda^{c} = \lambda^{c}(t).$$

Let, for example, red[redacted] is plot blue the preventive attack at some torque/moment t = 0, but blue, caught unaware, answer no counteraction to torque/moment t = τ, into which they introduce into action all their forces. Then in the equations of the dynamics of

combat

$$\left.\begin{aligned}\frac{dm_1^{\kappa}}{dt} &= -\lambda^{c} m_1^{c},\\ \frac{dm_1^{c}}{dt} &= -\lambda^{\kappa} m_1^{\kappa}\end{aligned}\right\} \qquad (7.2)$$

it is necessary to set/assume value $\lambda^c(t)$ by equal to zero to torque/moment t = τ and constant (equal to $\bar{\lambda}^c = \text{const}$) with t $\geqslant$ τ (Fig. 6.34).

So will be matter, if the side, subject to preventive attack, in no way answers enemy fire/light to torque/moment τ, but into torque/moment τ to section/shear are introduced into breakage all its forces.

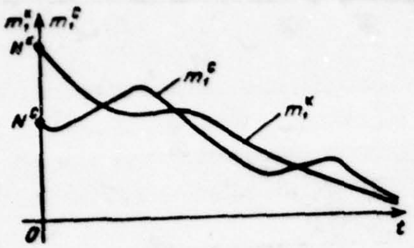

Fig. 6.33.

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It is more natural than to suppose that blue, subjected to attack, will begin gradually to mobilize and to introduce into action its forces. This I can take into account five- after all by the introduction to certain **alternating/variable** efficient rate of fire:

$$\lambda^c(t) = \tilde{\lambda}^c \varphi(t),$$

where $\tilde{\lambda}^c$ - nominal efficient rate of fire which will be reached after the termination of mobilization; $\phi(t)$ - certain increasing from 0 to 1 function (Fig. 6.35). Solving the equations of combat dynamics with the specific form of the function $\phi(t)$, it is possible to take into account the effect of the rate of mobilization on course and outcome of battle.

It can seem on first glance that the account to the mobilization of forces must be produced by the same methods, as the account of the input/introduction of reserves; but this is not so. The nonmobilized

up to torque/moment τ forces are found on the territory, subjected to the fire/light of enemy, having still entered into action; fresh forces (reserves) begin to undergo fire/light only with the torque/moment of input/introduction. Therefore mobilization and the input/introduction of reserves are considered differently: the first - by alternating/variable rate of fire, and the second - by an additional member in the right side of the equation.

Employing analogous procedure it is possible to take into account in the equations of combat dynamics and the limitedness of ammunition. Let us suppose that first that the ammunition of each combat unit is rigidly with it connected (it cannot be transmitted by other) and it is destroyed together with combat unit during its damage/defeat. Let the ammunition of each combat unit be red[redacted] calculated to time $\tau^{\text{к}}$, and blue - on τ^{o}. it is obvious,

$$\tau^{\text{к}} = k^{\text{к}}/\lambda_0^{\text{к}}; \quad \tau^{c} = k^{c}/\lambda_0^{c},$$

where k^{k} - supply of projectiles, available at each combat units red[redacted], k^o - the corresponding supply of blue, $\lambda_0^{\text{к}}(\lambda_0^{c})$ - the average rates of fire (not effective) one combat unit red[redacted] (blue).

In order to take into account the limitedness of ammunition, are sufficient to assume after torque/moment $\tau^{\text{к}}$ for red[redacted] and τ^{c} for blue efficient rates of fire equal to zero:

$$\lambda^{x}=0 \text{ with } t>\tau^{x};$$

$$\lambda^{c}=0 \text{ with } t>\tau^{c}.$$

Somewhat differently will be considered the limitedness of ammunition, if there is common/general/total for all units storage of ammunition supplies, reliably defended from the fire/light of enemy and which supplies with ammunition all units, the sides above another is expressed weaker, the decrease of numbers occurs more slowly.

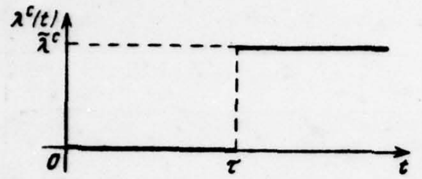

Fig. 6.34.

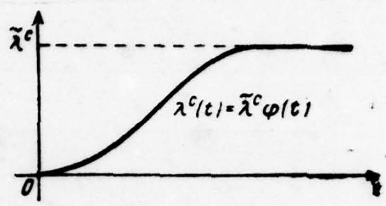

Fig. 6.35.

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Under conditions of model B (just as A) they can be, moreover by the same methods, are taken into acccunt the further factors: preventive attack, the addition/completion of forces, mobilization, etc.

9. *Model into the account to the activity of exploration and of the control system of breakage.*

In § 6 and 8 we considered two limiting cases of organizing the breakage: ideal organizations (model A) and poor organizations (model B).

In real reality the matter is not as good as in the first case,

but also not as it is badly/poor as in the second.

Transfer of fire from the affected unit to that nonafflicted is produced not instantly as in model A, but all the same it is produced. In real reality there are delays in transfer of fire, connected with delay obtainings of information about damage to target, and also with the noninstantaneous transmission of this information on the component/links of the control system of breakage. However, these delays are not so/such great so that would be obtained the schematic of model B, with its shooting "blindly" at all target/purposes - both affected and nonafflicted.

In this paragraph we will construct the unified model of breakage - model C, with respect to which previously introduced models A and B: are special cases. In model C, are considered such factors as activity of exploration and the degree of the efficiency of control of breakage.

Let us consider following model of breakage. Occurs the breakage of two groupings: K (is red[redacted]) and S (blue), that consist each of the uniform combat units in quantities N^K and N^C. The efficient rate of fire of one combat unit redder is equal to λ^K, blue - λ^C. Each combat unit redder can be located in the following states:

K_0 - is not reconnoitred,

K_1-is reconnoitred, but still not fired upon,

K_2 - Fired upon but is not still struck,

K_3-is struck, but this still nonl discovered; bombardment is continued,

K_4-is struck, this is discovered, but bombardment still is not removed,

K_5-is struck, bombardment removed.

The analogous states of the combat unit of blue let us designate C_0, C_1, C_2, C_3, C_4, C_5.

After bombardment from the affected unit it is removed, fire/light is transferred to any other from reconnoitred unity, which are found under bombardment, i.e., in states from the second on the fourth.

The graph/count of the states of the elements of system, divided into two subgraph K and C, is shown on Fig. 6.36.

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Let us **designate, as ever,** $X_0^{\kappa}, X_1^{\kappa}, \ldots, X_5^{\kappa}, X_0^{c}, X_1^{c}, \ldots, X_5^{c}$ - number of **states:** $m_0^{\kappa}, m_1^{\kappa} \ldots, m_5^{\kappa}, m_0^{c}, m_1^{c}, \ldots, m_5^{c}$ - corresponding **average** num**bers:** $\lambda_{ij}^{\kappa}, \lambda_{ij}^{c}$ - intensity of flow of the events, which translate combat **unit of Reds** (blue) from **state into** state. Let us determine these intensities, beginning **with** λ_{01}^{κ}. **Let us** consider that the transition of combat unit redder of the state K_0 (it is not reconnoitred) into state K_1 (it is reconnoitred) it occurs under the action of the flow of successful explorations of the Blues (under the flew of successful reconnaissances it is understood the flow of the events, which consist of the detection still of the not reconnoitred unit). It is obvious, this intensity depends on intensity and success of the reconnaissance actions of blue (flights of reconnaissance aviation, the search of reconnaissance parties, etc.). Let us designate the intensity of flow of the successful explorations of blue, that is necessary **per** each still not reconnoitred combat unit red[redacted], through $\lambda^{\kappa}_{\text{разв}}$; analogous designation for the intensity of successful explorations will red[redacted] be $\lambda^{c}_{\text{разв}}$. **Thus** we find the intensities of flow of the events, which translate **one** combat unit red[redacted] (analogously - blue) from state "it is not reconnoitred" into state "it is separated, but still it is not fired";

$$\lambda_{01}^{\kappa} = \lambda^{c}_{\text{разв}}; \quad \lambda_{01}^{c} = \lambda^{\kappa}_{\text{разв}}. \quad (9.1)$$

Let us note that both intensities $\lambda_{01}^{к}$, λ_{01}^{c} can be both dependent and independent on the general state, in which is located the grouping (i.e. from the numbers of states). This is caused by the fact, does function exploration autonomously, or reconnaissance resources are selected from the composition of grouping itself and thus they are translated from combat units in auxiliary.

It can render/show also that the intensity of flow of successful explorations depends on that, how much remained in the composition of the group of the enemy of the unexplored units. Thus, depending on the conditions of breakage, the parameters $\lambda_{\text{разв}}^{к}$ and $\lambda_{\text{разв}}^{c}$ can in one or another manner depend on the average numbers of states or not depend on them. We will not make more precise, which of these cases occurs, but simply let us designate $\lambda_{\text{разв}}^{c}$ - intensity of flow of the successful explorations of blue, which undergoes each still not dilute combat unit red[redacted] ($\lambda_{\text{разв}}^{к}$ - vice versa).

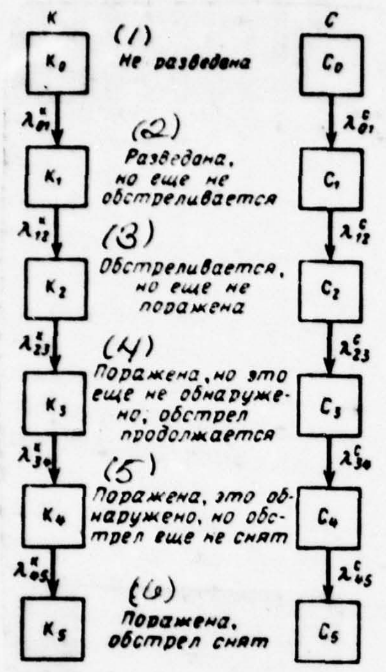

Fig. 6.36.

Key: (1). It is not reconnoitred. (2). It is reconnoitred, but still it is not fired. (3). It is fired, but is not still struck. (4). It is struck, but this is not still not discovered; bombardment is continued. (5). It is struck, this is discovered, but bombardment is not still removed. (6). It is struck, bombardment removed.

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For calculation $\lambda^{c}_{\text{разв}}$ it is necessary to determine, how often for the time unit in combat unit this (arbitrarily selected) unexplored

region red appears the prospector of blue (for example, the aircraft of reconnaissance aviation), and to multiply this number by probability that the unit will be discovered by prospector ($\lambda^{\text{к}}_{\text{разв}}$. it is located so).

Let us find the intensity of flow of the events, which translate the unit from state K_1 into K_2. The corresponding event lies in the fact that the reconnoitred unit is placed under bombardment. The intensity of flow of events can be defined, as value, reciprocal to mean delay time in setting under the bombardment of reconnoitered combat unit red. This time depends on the degree of perfection and operating speed of the system of control of blue; let us designate it $\bar{t}_{\text{пост}}$. Then

$$\lambda^{\text{к}}_{12} = 1/\bar{t}^{\text{c}}_{\text{пост}}. \qquad (9.2)$$

analogously

$$\lambda^{\text{c}}_{12} = 1/\bar{t}^{\text{к}}_{\text{пост}}. \qquad (9.3)$$

Let us find the intensity of flow of events, which translates the combat unit from state K_2 (it is fired upon, but is not still struck) into state K_3 (it is struck). This is the flow of the successful (damaging) shots of blue, that is necessary per one combat unit red[redacted], that is found in state K_2. From what does store/add up this flow? From the side C, participate in bombardment all unit, which are found in states C_0, C_1, C_2; their number is equal

$$X_0^{\text{c}} + X_1^{\text{c}} + X_2^{\text{c}}.$$

Each of them makes λ^{c} successful shots per unit time. According to condition, these shots evenly are distributed between all combat units redder, that are found in states K_2, K_3, K_4. Their number equally

$$X_2^{\text{к}}+X_3^{\text{к}}+X_4^{\text{к}},$$

means for each of them it comes the flow of successful shots with the intensity

$$\frac{\lambda^{c}(X_0^{c}+X_1^{c}+X_2^{c})}{X_2^{\text{к}}+X_3^{\text{к}}+X_4^{\text{к}}}.$$

These intensity as we know that it is necessary to still multiply by function $R(X_2^{\text{к}}+X_3^{\text{к}}+X_4^{\text{к}})$, that turning into zero, when $X_2^{\text{к}}+X_3^{\text{к}}+X_4^{\text{к}}=0$, i.e., there are no units of Reds which it would be possible to fire (see formula (4.4) §4). Using designation $\frac{R(x)}{x}=\rho(x)$ (see formula (4.5) §4), we obtain the intensity of flow of events, which translates the combat unit red[redacted]r from state K_2 into K_3:

$$\lambda_{23}^{\text{к}}=\lambda^{c}(X_0^{c}+X_1^{c}+X_2^{c})\cdot\rho(X_2^{\text{к}}+X_3^{\text{к}}+X_4^{\text{к}}). \qquad (9.4)$$

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Let us analogously find the intensity of flow of events, which translates the combat unit blue from state C_2 into C_3:

$$\lambda_{23}^{c}=\lambda^{\text{к}}(X_0^{\text{к}}+X_1^{\text{к}}+X_2^{\text{к}})\cdot\rho(X_2^{c}+X_3^{c}+X_4^{c}). \qquad (9.5)$$

Let us determine the intensity of flow of events, which translates the unit from state K_3 into K_4; this the intensity of flow of the successful control explorations of blue, supply/delivering to

them the information about damage/defeat fired units. In the general case the intensity of flow of control explorations does not coincide with the intensity of flow of the explorations, directed toward the detection of new targets; they even can be realized by different forces. Let us designate the intensity of flow of the control explorations of blue $\lambda^{c}_{\text{конт}}$ (this value can be counted as reverse to the mean time, which separate/liberates the torque/moment of damage to target from the torque/moment of the detection of it by control exploration). We have

$$\lambda^{\kappa}_{34} = \lambda^{c}_{\text{конт}} \tag{9.6}$$

and it is analogous

$$\lambda^{c}_{34} = \lambda^{\kappa}_{\text{конт}}. \tag{9.7}$$

Intensities $\lambda^{\kappa}_{\text{конт}}, \lambda^{c}_{\text{конт}}$ can be both dependent and independent of the average numbers of states.

It remains to determine the intensity of flow of events, which translates the unit from K_4 into K_4. This intensity can be considered the value, reciprocal to the mean duration of transmission of order about removal/taking of fire/light from unit after its damage/defeat recorded by the exploration of blue. Let us designate this mean time $\bar{t}^{c}_{\text{сн}}$ (is analogous for red[redacted] $\bar{t}^{\kappa}_{\text{сн}}$). We will obtain:

$$\lambda^{\kappa}_{45} = 1/\bar{t}^{c}_{\text{сн}}, \tag{9.8}$$

$$\lambda^{c}_{45} = 1/\bar{t}^{\kappa}_{\text{сн}}. \tag{9.9}$$

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Using the graph/count of states (Fig. 6.36), by intensities (9.1)-(9.8) and by applying the principle of quasi-regularity, let us register the equations of combat dynamics in the form:

$$\left.\begin{aligned}
&\frac{dm_0^\kappa}{dt}=-\lambda^c_{\text{разв}}m_0^\kappa,\\
&\frac{dm_1^\kappa}{dt}=-\frac{1}{\bar t^c_{\text{пост}}}m_1^\kappa+\lambda^c_{\text{разв}}m_0^\kappa,\\
&\frac{dm_2^\kappa}{dt}=-\lambda^c(m_0^c+m_1^c+m_2^c)\,\rho\,(m_2^\kappa+m_3^\kappa+m_4^\kappa)\,m_2^\kappa+\\
&\qquad\qquad+\frac{1}{\bar t^c_{\text{пост}}}m_1^\kappa,\\
&\frac{dm_3^\kappa}{dt}=-\lambda^c_{\text{конт}}m_3^\kappa+\lambda^c\left(m_0^c+m_1^c+m_2^c\right)\times\\
&\qquad\times\rho\left(m_2^\kappa+m_3^\kappa+m_4^\kappa\right)m_2^\kappa,\\
&\frac{dm_4^\kappa}{dt}=-\frac{1}{\bar t^c_{\text{сн}}}m_4^\kappa+\lambda^c_{\text{конт}}m_3^\kappa,\\
&\frac{dm_0^c}{dt}=-\lambda^\kappa_{\text{разв}}m_0^c,\\
&\frac{dm_1^c}{dt}=-\frac{1}{\bar t^\kappa_{\text{пост}}}m_1^c+\lambda^\kappa_{\text{разв}}m_0^c,\\
&\frac{dm_2^c}{dt}=-\lambda^\kappa(m_0^\kappa+m_1^\kappa+m_2^\kappa)\,\rho\,(m_2^c+m_3^c+m_4^c)\,m_2^c+\\
&\qquad+\frac{1}{\bar t^\kappa_{\text{пост}}}m_1^c,\\
&\frac{dm_3^c}{dt}=-\lambda^\kappa_{\text{конт}}m_3^c+\lambda^\kappa\left(m_0^\kappa+m_1^\kappa+m_2^\kappa\right)\times\\
&\qquad\times\rho\left(m_2^c+m_3^c+m_4^c\right)m_2^c,\\
&\frac{dm_4^c}{dt}=-\frac{1}{\bar t^\kappa_{\text{сн}}}m_4^c+\lambda^\kappa_{\text{конт}}m_3^c.
\end{aligned}\right\}\quad(9.10)$$

Equations for m_5^κ, m_5^c are reject/thrown, since for any moment t

$$\left.\begin{aligned}
m_5^\kappa&=N^\kappa-(m_0^\kappa+m_1^\kappa+m_2^\kappa+m_3^\kappa+m_4^\kappa),\\
m_5^c&=N^c-(m_0^c+m_1^c+m_2^c+m_3^c+m_4^c).
\end{aligned}\right\}\quad(9.11)$$

moreover as a rule, us do not interest the number of units, affected and not fired (and thereby of the leaving from number both active and

passive elements of system).

Differential equations (9.10) at any values of the entering in them parameters can be solved numerically (in machine or by hand). Initial conditions depend on the tactical situation which it is required to trace. For example, if at the beginning of combat operations **some portion/fraction** of combat units is already reconnoitred (α^K for red[redacted] and α^c for blue), then initial conditions will be:

$$
\begin{aligned}
&t=0, \quad m_0^K = N^K(1-\alpha^K); \quad m_1^K = N^K\alpha^K;\\
&m_2^K = m_3^K = m_4^K = m_5^K = 0;\\
&m_0^c = N^c(1-\alpha^c); \quad m_1^c = N^c\alpha^c;\\
&m_2^c = m_3^c = m_4^c = m_5^c = 0.
\end{aligned}
$$

The examined by us model of breakage **C** is more common/general/total than previously examined models **A** and **B** which escape/ensue from model **C** as special cases.

It is real/actual, if we consider at the initial moment all units reconnoitred, and the time, necessary for the detection of the fact of damage to target and for the transmission of information about this on all component/links of the control system, equal to zero - will be obtained model **A** (in this case three first states will merge in one: "unit is not struck", and three latter - also in one: "unit is struck").

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Model B will be obtained, if we also consider at the initial moment all units reconnoitred, but the time, necessary for obtaining and transmission of information, to assume equal to infinity.

The equations of model C, which include, besides purely combat operations, even reconnaissance, and also considering the degree of the efficiency of control of breakage, make it possible to solve the problems, connected as with the "value of information" in the process of developing the combat operations.

Let us note additionally that in the equations of model C, as for models A and B, it is possible to easily take into account all the further factors, which accompany combat operations (preventive attack, the addition/completion of forces, the rate of mobilization, etc.).

10. Account of the restoration/reduction of the units in the course of combat operations.

Until now, examining the equations of combat dynamics, we

assumed that the affected combat unit finally retires from system. Generally speaking, this nct always so, in the case when the duration cf combat operations is great in comparison with the time, required to the repair of unit, can arise speech abcut the account of the restoration/reduction of the units in the course of combat operations. The same task appears also in cases when the combat action of unit during its "damage/defeat" ceases only temporarily (for example, due to the effect of interferences). V0 all these cases the unit, temporarily failing, it can in (generally speaking, random) for a while again enter into system.

To take into account this restoration/reduction in the equations of combat dynamics does not represent work. Let us show how this to do, based on the simplest example, close according to diagram to model A (in the case of necessity analcgously it is possible to take into account the restoration/reduction of the units in any other model).

Let in combat participate both sides K and C in the composition N^K and N^C of combat units; each of them can be in one of the states:

K_1 (C_1) - functions normally,

K_2 (C_2) is injured, is repaired,

K_3 (C_3) is struck finally, to repair it is not subject.

Graph/count of states, who falls into subgraphs C and K, shown on Fig. 6.37.

The organization of breakage is assumed following.

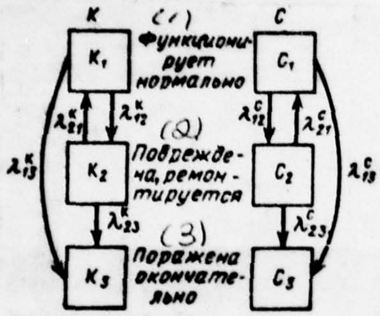

Fig. 6.37.

Key: (1). It functions normally. (2). It is injured, it is overhauled. (3). It is struck finally.

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1. Each combat unit of any side can conduct fire/light on any combat unit of enemy.

2. Fire/light sighting, each shot can damage only TU unit along which it is directed.

3. Fire/light evenly is distributed between all finally not affected units both functioning and overhauled.

4. Damaged unit fire/light does not conduct.

5. During final damage of unit, fire/light from it immediately is remove/taken and is transferred to other, still that not affected.

All flows of events, as ever let us consider Poisson. Each combat unit redder produces the flow of shots with intensity $\lambda^{к}$, blue - with intensity $\lambda^{с}$. The shot, directed along intact/uninjured/undamaged unit redder, it injures it (it translates from K_1 into K_2) with probability $\mathcal{P}_1^{к}$ and it finally strikes it with probability $\mathcal{P}_2^{к}$. The shot, directed along the already damaged unit, strikes it finally (it translates into state K_3) with probability $\mathcal{Q}_3^{к}$; otherwise the state of the unit does not vary. Analogous data for the combat unit of blue will be $\mathcal{P}_1^{с}$, $\mathcal{P}_2^{с}$, $\mathcal{Q}_3^{с}$.

The mean time of the repair: (restoration/reduction) the damaged combat unit redder is equal to $\bar{\tau}^{к}_{рем}$, blue - $\bar{\tau}^{с}_{рем}$.

Let us write the equations of the dynamics of average for this system. Let us introduce the usual designations of numbers and average numbers of states:

$$X_1^{к},\ X_2^{к},\ X_3^{к},\ X_1^{с},\ X_2^{с},\ X_3^{с},$$
$$m_1^{к},\ m_2^{к},\ m_3^{к},\ m_1^{с},\ m_2^{с},\ m_3^{с}$$

it is expressed all the intensities $\lambda^{к}_{ij}$, $\lambda^{с}_{ij}$ through the assigned parameters and the numbers of states.

Let us find intensity λ_{12}^{κ}. In all on side K at torque/moment t it shoots X_1^c the units of blue; each of them produces, on the average, λ^c shots per unit time (in this case, it is simple of "shots", but not "successful shots"). These shots evenly are distributed between all functioning and restorable units red[redacted]. The fired undamaged unit with probability $\mathcal{P}_2^{\kappa}$ is damaged, with probability $\mathcal{P}_3^{\kappa}$ completely it is derive/concluded from system. It is obvious, per each unit in state K_1, it comes in the unit of time on the average

$$\frac{\lambda^c X_1^c \mathcal{P}_2^{\kappa}}{X_1^{\kappa}+X_2^{\kappa}}$$

of the "damaging" shots; this intensity of flow damaging shots must be multiplied by function $R(X_1^{\kappa}+X_2^{\kappa})$(see formula (4.4) §4), that turns into zero, when there is not one unit which it is possible to fire upon. We will obtain:

$$\lambda_{12}^{\kappa}=\frac{\lambda^c X_1^c \mathcal{P}_2^{\kappa}}{X_1^{\kappa}+X_2^{\kappa}} R(X_1^{\kappa}+X_2^{\kappa}), \qquad (10.1)$$

where correction factor $R(X_1^{\kappa}+X_2^{\kappa})$ for the initial stages of breakage it is possible not to consider (to set/assume equal to unit).

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We will analogously obtain the intensity of flow of the events, which translate the unit red[redacted] from state K_1 into K_3 (it is struck completely):

$$\lambda_{13}^{\kappa}=\frac{\lambda^c X_1^c \mathcal{P}_3^{\kappa}}{X_1^{\kappa}+X_2^{\kappa}} R(X_1^{\kappa}+X_2^{\kappa}). \qquad (10.2)$$

Intensity of flow of events, which translates the unit redder of K_2 into K_3:

$$\lambda_{23}^{\kappa} = \frac{\lambda^{c} X_1^{c} \mathcal{Q}_3^{\kappa\kappa}}{X_1^{\kappa} + X_2^{\kappa}} R(X_1^{\kappa} + X_2^{\kappa}). \qquad (10.3)$$

Finally, the value of the intensity of flow of events (restoration/reductions), that translates the unit from K_2 into K_1, is reverse to the mean time of the repair:

$$\lambda_{21}^{\kappa} = \frac{1}{\bar{t}_{\text{рем}}^{\kappa}}. \qquad (10.4)$$

Passing in the expressions of intensities from function R(x) to function $\rho(x) = R(x)/x$, we will obtain:

$$\left.\begin{aligned} &\lambda_{21}^{\kappa} = 1/\bar{t}_{\text{рем}}^{\kappa}, \\ &\lambda_{12}^{\kappa} = \lambda^{c} X_1^{c} \mathcal{P}_2^{\kappa}\, \rho(X_1^{\kappa} + X_2^{\kappa}), \\ &\lambda_{13}^{\kappa} = \lambda^{c} X_1^{c} \mathcal{P}_3^{\kappa}\, \rho(X_1^{\kappa} + X_2^{\kappa}), \\ &\lambda_{23}^{\kappa} = \lambda^{c} X_1^{c} \mathcal{Q}_3^{\kappa}\, \rho(X_1^{\kappa} + X_2^{\kappa}), \end{aligned}\right\} \qquad (10.5)$$

and it is analogous for blue:

$$\left.\begin{aligned} &\lambda_{21}^{c} = 1/\bar{t}_{\text{р м}}^{c}, \\ &\lambda_{12}^{c} = \lambda^{\kappa} X_1^{\kappa} \mathcal{P}_2^{c}\, \rho(X_1^{c} + X_2^{c}), \\ &\lambda_{13}^{c} = \lambda^{\kappa} X_1^{\kappa} \mathcal{P}_3^{c}\, \rho(X_1^{c} + X_2^{c}), \\ &\lambda_{23}^{c} = \lambda^{\kappa} X_1^{\kappa} \mathcal{Q}_3^{c}\, \rho(X_1^{c} + X_2^{c}). \end{aligned}\right\} \qquad (10.6)$$

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Taking into account the graph of Fig. 6.37 and of intensities (10.5), (10.6), using the principle of quasi-regularity, let us write the system of equations of combat dynamics with restoration of the units:

$$\left.\begin{aligned}
\frac{dm_1^{\kappa}}{dt} &= -\lambda^{c} m_1^{c} (\mathcal{P}_2^{\kappa} + \mathcal{P}_3^{\kappa})\, \rho\, (m_1^{\kappa} + m_2^{\kappa})\, m_1^{\kappa} + \\
&\quad + \frac{1}{\bar{t}^{\kappa}_{\text{рем}}} m_2^{\kappa}, \\
\frac{dm_2^{\kappa}}{dt} &= -\frac{1}{\bar{t}^{\kappa}_{\text{рем}}} m_2^{\kappa} - \lambda^{c} m_1^{c}\, \mathcal{Q}_3^{\kappa}\, \rho\, (m_1^{\kappa} + m_2^{\kappa})\, m_2^{\kappa} + \\
&\quad + \lambda^{c} m_1^{c}\, \mathcal{P}_2^{\kappa}\, \rho\, (m_1^{\kappa} + m_2^{\kappa})\, m_1^{\kappa}, \\
\frac{dm_1^{c}}{dt} &= -\lambda^{\kappa} m_1^{\kappa} (\mathcal{P}_2^{c} + \mathcal{P}_3^{c})\, \rho\, (m_1^{c} + m_2^{c})\, m_1^{c} + \\
&\quad + \frac{1}{\bar{t}^{\kappa}_{\text{рем}}} m_2^{c}, \\
\frac{dm_2^{c}}{dt} &= -\frac{1}{\bar{t}^{c}_{\text{рем}}} m_2^{c} - \lambda^{\kappa} m_1^{\kappa}\, \mathcal{Q}_3^{c}\, \rho\, (m_1^{c} + m_2^{c})\, m_2^{c} + \\
&\quad + \lambda^{\kappa} \mathcal{P}_2^{c}\, \rho\, (m_1^{c} + m_2^{c})\, m_1^{c}.
\end{aligned}\right\} \quad (10.7)$$

As concerns m_3^{κ}, m_3^{c}, that for any moment t

$$m_3^{\kappa} = N^{\kappa} - (m_1^{\kappa} + m_2^{\kappa}),$$
$$m_3^{c} = N^{c} - (m_1^{c} + m_2^{c});$$

moreover these state us, as a rule, they do not interest.

The system of nonlinear differential equations (10.7) for any concrete/specific/actual values of the entering it parameters can be solved numerically (in machine or by hand). Initial conditions, as ever, are assigned on the basis of tactical considerations. For example, if us it interests the behavior of system soon after discovery/opening of combat operations, then it is possible to take

the initial conditions:

$$t=0;\quad m_1^{\kappa}=N^{\kappa};\quad m_2^{\kappa}=m_3^{\kappa}=0,$$
$$m_1^{c}=N^{c};\quad m_2^{c}=m_3^{c}=0.$$

However, us can interest and the ability of one or the other side "to be chosen" from the difficult situation, when at the initial moment a considerable quantity of its units is injured.

Let us note that for the initial stages of breakage, then the average numbers of intact/uninjured/undamaged and overhauled units $m_1^{\kappa}+m_2^{\kappa}$, $m_1^{c}+m_2^{c}$ are still sufficiently great, correction factors $R(m_1^{\kappa}+m_2^{\kappa})$, $R(m_1^{c}+m_2^{c})$ are converted into unit, and that means that $\rho(m_1^{\kappa}+m_3^{\kappa})$ it is possible to replace by $1/(m_1^{\kappa}+m_3^{\kappa})$, and $\rho(m_1^{c}+m_3^{c})$ by $1/(m_1^{c}+m_3^{c})$.

During the solution of problem, we for simplicity assumed that the fire/light is distributed evenly between all finally not affected units - both damaged and intact/uninjured/undamaged. However, this completely not is compulsory: it is easy to take into account unequal distribution of fire between those and others.

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For this, it suffices to multiply the appropriate intensities of flow of shots per some coefficients, high units for those cell/elements which are fired preferably, and smaller units - for the others: these coefficients can be both the constants and variables. To the

methodology of the account to the nonuniformity of distribution of fire, we will be introduced in following paragraph.

11. Equations of combat dynamics for heterogeneous units. Functions of distribution of fire.

Until now, we examined only the groupings, consisting of uniform combat units. Does not represent the work to write the equations of combat dynamics, also, for the case when the combat units, entering the grouping, are heterogeneous. Let us demonstrate this again based on the example of the simplest model, close in form of organization to model A, but by the differing from it diversity of units.

Let occur the breakage between two groupings K and C, grouping K consisting of the heterogeneous combat units of types k and $\varkappa$, but grouping C - from the heterogeneous combat units of types c and γ (Fig. 6.38). A quantity of each type combat units is equal respectively $N^k, N^\varkappa, N^c, N^\gamma$. Each combat unit can be in one of two states: it is not struck, struck. Shooting is conducted only on the nonafflicted units (obtaining and account to information are instantaneous as in model A).

The graph/count of the states of unit is shown on Fig. 6.39 - he falls into four subgraph: k and $\varkappa$, c and γ (according to the

number of types of units). As let us usually designate the numbers of states and the average numbers of states respectively

$$X_1^k,\ X_2^k,\ X_1^x,\ X_2^x,\ X_1^c,\ X_2^c,\ X_1^v,\ X_2^v,$$
$$m_1^k,\ m_2^k,\ m_1^x,\ m_2^x,\ m_1^c,\ m_2^c,\ m_1^v,\ m_2^v.$$

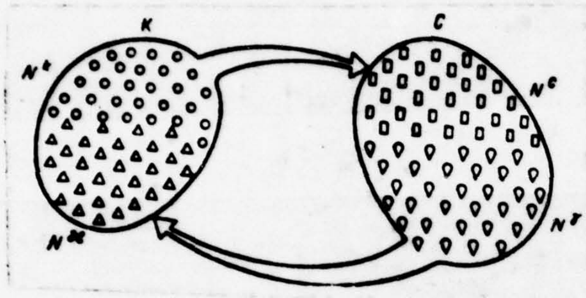

Fig. 6.38.

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In order to determine the intensities of flow of the events, which translate the units from state into state, it is necessary to assign some rule of distribution of fire between the units of different types. This rule it will prescribe at each moment of time t some portion/fraction of the available available combat devices each type to direct on units of first type enemy, and all others - on second type unity.

Let us introduce designation for the functions, which describe this distribution (let us agree the type of the shooting unit to place at letter with the first index, and fired upon - by the second). Let us introduce the designations:

$\alpha_{k,c}(t)$ - portion/fraction of the nonafflicted combat units of the type k whose fire/light for torque/moment t is directed on combat units of the type c,

$\alpha_{k,\gamma}(t)$ - a portion/fraction of the nonafflicted combat units of the type k whose fire/light for torque/moment t is directed on combat units of the type γ.

It is obvious that since at the any moment of time fire/light conduct all the capable of this unit,

$$\alpha_{k,\gamma}(t)=1-\alpha_{k,c}(t).$$

Let us analogously designate

$$\alpha_{\varkappa,c}(t),\ \alpha_{c,k}(t),\ \alpha_{\gamma,k}(t)$$

the portion/fraction of the nonafflicted combat units of types $\varkappa$, c, γ respectively whose fire/light for torque/moment t is directed on unity of types c, k, k respectively, and let us name four functions

$$\alpha_{k,c}(t),\ \alpha_{\varkappa,c}(t),$$
$$\alpha_{c,k}(t),\ \alpha_{\gamma,k}(t)$$

the functions of distribution of fire.

Besides the functions of distribution of fire, it is necessary to assign also the characteristics of the efficiency of the fire/light of different units on different target/purposes. Let us designate:

$$\lambda^{\varkappa},\ \lambda^{\varkappa},\ \lambda^{c},\ \lambda^{\gamma} \qquad (11.1)$$

the intensities of flow of the shots of the corresponding combat units. Furthermore, let us designate:

$\mathcal{P}_{k,c}$ - kill probability of a combat unit of the type c with one shot on it of combat unit of the type k,

$\mathcal{P}_{k,y}$; $\mathcal{P}_{x,c}$; $\mathcal{P}_{x,y}$; $\mathcal{P}_{c,k}$; $\mathcal{P}_{c,x}$; $\mathcal{P}_{y,k}$; $\mathcal{P}_{y,x}$ - is analogous.

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In these designations again the index of shooting units - to the left, fired upon - to the right.

Multiplying the intensities of flow of shots to the appropriate kill probabilities, we will obtain the efficient rates of fire:

$$\left.\begin{array}{ll} \lambda_{k,c}=\lambda^{k}\mathcal{P}_{k,c}; & \lambda_{k,y}=\lambda^{k}\mathcal{P}_{k,y}; \\ \lambda_{x,c}=\lambda^{x}\mathcal{P}_{x,c}; & \lambda_{x,y}=\lambda^{x}\mathcal{P}_{x,y}; \\ \lambda_{c,k}=\lambda^{c}\mathcal{P}_{c,k}; & \lambda_{c,x}=\lambda^{c}\mathcal{P}_{c,x}; \\ \lambda_{y,k}=\lambda^{y}\mathcal{P}_{y,k}; & \lambda_{y,x}=\lambda^{y}\mathcal{P}_{y,x}. \end{array}\right\} \quad (11.2)$$

Now it is possible to find the intensities of all flows of events for the graph of Fig. 6.39.

Let us determine λ_{12}^{k}. For this, let us find the average number of successful shots, which is necessary per one combat unit of the type k for time unit. In all on units of the type k for unit of time it comes

$$\alpha_{c,k}\lambda_{c,k}X_1^{c}+\alpha_{y,k}\lambda_{y,k}X_1^{y}$$

successful shots. This number must be divided into the number $X_1^к$ of the combat units in state k_1 and multiplied by correction factor $R(X_1^к)$. Hence, passing from function R(x) to function $\rho(x) = R(x)/x$, we will obtain:

$$\lambda_{12}^{к} = (\alpha_{с.к}\lambda_{с.к}X_1^{с} + \alpha_{у.к}\lambda_{у.к}X_1^{у})\,\rho(X_1^{к}). \qquad (11.3)$$

It is analogous

$$\lambda_{12}^{н} = (\alpha_{с.н}\lambda_{с.н}X_1^{с} + \alpha_{у.н}\lambda_{у.н}X_1^{у})\,\rho(X_1^{н}), \qquad (11.4)$$
$$\lambda_{12}^{с} = (\alpha_{к.с}\lambda_{к.с}X_1^{к} + \alpha_{н.с}\lambda_{н.с}X_1^{н})\,\rho(X_1^{с}), \qquad (11.5)$$
$$\lambda_{12}^{у} = (\alpha_{к.у}\lambda_{к.у}X_1^{к} + \alpha_{н.у}\lambda_{н.у}X_1^{н})\,\rho(X_1^{у}). \qquad (11.6)$$

These expressions are possible to simplify somewhat if we join the *functions of distribution of fire with efficient rates of fire* and to designate:

$$\left.\begin{array}{ll} \beta_{к.с} = \alpha_{к.с}\lambda_{к.с}, & \beta_{к.у} = \alpha_{к.у}\lambda_{к.у}, \\ \beta_{н.с} = \alpha_{н.с}\lambda_{н.с}, & \beta_{н.у} = \alpha_{н.у}\lambda_{н.у}, \\ \beta_{с.к} = \alpha_{с.к}\lambda_{с.к}, & \beta_{с.н} = \alpha_{с.н}\lambda_{с.н}, \\ \beta_{у.к} = \alpha_{у.к}\lambda_{у.к}, & \beta_{у.н} = \alpha_{у.н}\lambda_{у.н}. \end{array}\right\} \qquad (11.7)$$

Newly introduced functions (11.7) can be named the distribution functions of efficiency.

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Taking into account these designations and formulas (11.2)-(11.6) it is possible to register differential equations for the average

numbers of states (equaticn of combat dynamics) in the form:

$$\left.\begin{aligned}\frac{dm_1^k}{dt} &= -(\beta_{c.k} m_1^c + \beta_{v.k} m_1^v) R(m_1^k),\\ \frac{dm_1^x}{dt} &= -(\beta_{c.x} m_1^c + \beta_{v.x} m_1^v) R(m_1^x),\\ \frac{dm_1^c}{dt} &= -(\beta_{k.c} m_1^k + \beta_{x.c} m_1^x) R(m_1^c),\\ \frac{dm_1^v}{dt} &= -(\beta_{k.v} m_1^k + \beta_{x.v} m_1^x) R(m_1^v).\end{aligned}\right\} \quad (11.8)$$

The average numbers of remaining states (usually us not interesting) can be found from the conditicns:

$$\left.\begin{aligned} m_2^k = N^k - m_1^k; \quad m_2^x = N^x - m_1^x;\\ m_2^c = N^c - m_1^c; \quad m_2^v = N^v - m_1^v.\end{aligned}\right\} \quad (11.9)$$

Let us note that equation (11.8) for initial of the stages of the breakage, when correcticn factors $R(m_1^k), R(m_1^x), R(m_1^c), R(m_1^v)$ equal to unit, are linear equations (in the general case with variable coefficients). The solution of similar equations (in machine or by hand) difficulties does not represent.

Let us note that, using similar equations (number of heterogeneous cell/elements in which easy to increase), it is possible to not only approximately describe the course of combat operations during the assigned functicns of distribution of fire, but also to optimize control cf breakage, i.e., to find the most advantageous form of these functions.

In conclusion let us note that in similar type equations it is

possible to examine the dynamics of a change in the numbers not only of of combat units, but also any auxiliary (radar stations, conveying devices, etc.). It goes without saying that for all such units it is necessary to set/assume efficient rates of fire equal to zero.

12. Mixed type equations.

Until now, we described the processes, taking place in physical systems, either with the help of equations for the probabilities of the states (see Chapter 4 and 5), or with the help of the equations of the dynamics of average (Chapter 6), where the unknown functions are the average numbers of states. First type equations were applied when system was comparatively simple and its states - are comparatively scarce. Second type equations were specially intended for describing the processes, taking place in the systems, consisting of numerous cell/elements; for such systems we succeeded in finding not the probabilities of states, but, in the first place, the average numbers of states.

In practice are encountered the situations in which it is necessary to apply mixed type of equation. In these equations figure both the probabilities of states and the average numbers of states.

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This apparatus is applied, when system S, in which occurs the process, consists of different type cell/elements: scarce (unique) and numerous (associating), moreover the states of each are interrelated.

In the similar cases for cell/elements of the first type, it is possible to comprise the differential equations, in which unknown functions are the probabilities of states; for cell/elements of the second type - equation of the dynamics of average, where the unknown functions - average numbers of states. Such equations we will call mixed type equations. ¶ As an example let us consider system S, which consists of the large quantity N of uniform instruments of (elements) and one voltage regulator C, which performs the important function of the provision for a normal mode of the work of all instruments immediately. Both the stabilizer and separate instruments can go out of order (reject). The intensity of flow of the malfunctions of stabilizer depends from the number x of the working instruments:

$$\lambda^c = \varphi(x). \qquad (12.1)$$

The left the system stabilizer immediately begins to be overhauled; the mean time of the repair of stabilizer depends on number simultaneously with it the located in repair instruments y:

$$\bar{t}^c_{рем} = \psi(y). \qquad (12.2)$$

The intensity of flow of the malfunctions of each instrument with working stabilizer is equal to μ_c, with nonworking - $\mu_{\bar{c}}$. The refused instrument immediately begins to be overhauled; the mean time of the repair of instrument depends on that, is overhauled stabilizer and how many instruments are overhauled simultaneously. With the not overhauled stabilizer this time is equal to $f_c(y)$, with that overhauled - $f_{\bar{c}}(y)$, where y - number of simultaneously overhauled instruments, a f_c, $f_{\bar{c}}$ - some functions.

It is required to describe the process, which takes place in system, with the help of mixed type equations, in which unknown functions they will be:

- probability of states (for a stabilizer),

- average numbers of states (for instruments).

The methodology of the composition of such equations differs from already known to us the methodology of the composition of the equations of the dynamics of average. In fact, yell the composition of equations for the average numbers of states we they used the principle of quasi-regularity, being based on what the values of random variable X_i - the number of i state were close to their average value m_i, they are grouped around this average value. When,

in the system, "unique" cell/element is present, already there are no foundations for considering that this thus. In this case, typical will be another situation, when the distribution of the numbers of states of auxiliary cell/element takes the two-peak form as, for example, shown on Fig. 6.40.

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Along the axis of abscissa are plot/deposited the numbers of some state of auxiliary element, while along the axis of ordinates, - the corresponding probabilities. If concrete/specific/actually speech occurs about number X_1 of working cell/elements, then is the right group of the value (see Fig. 6.40) it corresponds to the operation of system in exact stabilizer, and left - with defective (it goes without saying that considering that the work of stabilizer it is favorable for instruments). If distribution is such, as in Fig. 6.40, then random variable will be sometimes close to the average value of left group, sometimes - to the average of right group, but virtually never it will be close to the "full/total/complete" average value of random variable which lie/rests somewhere between both groups. In such cases the principle of quasi-regularity is inapplicable.

Let us look, it is not possible with anything to replace this principle in order to nevertheless solve stated problem? It turns out

that it is possible really then that we sufficiently indefinitely called the "average value of one group" (in the case, when distribution is grouped in two places on segment from 0 to N) - this nothing else but the conditional mathematical expectation of random variable X_1 when the stabilizer works - for one group, or when the stabilizer does not work - for another.

Recall what conditional mathematical expectation is. The usual mathematical expectation of random variable X_1 (unconditional) is defined as sum

$$M[X_1] = \sum_{k=0}^{N} kp_k, \qquad (12.3)$$

where p_k - probability that random variable X_1, will take value of k:

$$p_k = P(X_1 = k), \quad (k = 0, 1, \ldots, N). \qquad (12.4)$$

Taking into account (12.4) formula (12.3) can be rewritten in the form:

$$M[X_1] = \sum_{k=0}^{N} k \cdot P(X_1 = k). \qquad (12.5)$$

Let us examine now any random event C (in application to our case - event, which consists of the fact that the stabilizer works. Let us determine conditionally the mathematical expectation of random variable X_1 under the condition of event C, after replacing in formula (12.5) of probability - by conditional probabilities:

$$M[X_1/C] = \sum_{k=0}^{N} k \cdot P(X_1 = k/C), \qquad (12.6)$$

where $P(X_1 = k/C)$ - conditional probability that the random value X_1 will take value of k, when occurs event C.

Fig. 6.40.

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Analogously will be written determination, also, for conditional mathematical expectations $M[X_1/\bar{C}]$, $M[X_2/C]$, $M[X_2/\bar{C}]$ (random variable X_2 - number of instruments of state of repair, $\bar{C}$ - event, which consists of the fact that the stabilizer is overhauled).

We convert formula (12.6) to another form. For this, we will use expression for the conditional probability of any event A when event C occurs:

$$P(A/C)=\frac{P(AC)}{P(C)}. \qquad (12.7)$$

Then formula (12.6) will take the form:

$$M[X_1/C]=\sum_{k=0}^{N}k\frac{P(C,\ X_1=k)}{P(C)}=\frac{1}{P(C)}\sum_{k=0}^{N}kP(C,X_1=k). \qquad (12.8)$$

Here $P(C,\ X_1 = k)$ indicates probability that occur both events: and C, and $X_1 = k$ (i.e. stabilizer works and random variable X_1 took value of k).

In order to simplify expression (12.8) let us introduce new random variable:

$$X_1^c = \begin{cases} X_1, \text{ if occurs event C,} \\ 0, \text{ if event C it does not occur.} \end{cases}$$

With the help of this random variable X_1^c conditional mathematical expectation $M[X_1/C]$ will be registered in the following manner:

$$M[X_1/C] = \frac{1}{P(C)} M[X_1^c]. \quad (12.9)$$

It is real/actual, for k ≠ 0

$$P(X_1^c = k) = P(C, X_1 = k);$$

therefore the mathematical expectation of random variable X_1^c will be registered as

$$M[X_1^c] = \sum_{k=1}^{N} k \cdot P(C, X_1 = k)$$

or, taking into account that the member of sum, who corresponds to k = 0, is equal to zero,

$$M[X_1^c] = \sum_{k=0}^{N} k \cdot P(C, X_1 = k).$$

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It is analogous, introducing into examination random variables

$$X_1^{\bar{c}} = \begin{cases} 0, & \text{if event C it occurs,} \\ X_1, & \text{if event C it does not occur (i.e. it occurs } \bar{C}\text{),} \end{cases}$$

$$X_2^{c} = \begin{cases} X_2, & \text{if event C occurs,} \\ 0, & \text{if event C it does not occur.} \end{cases}$$

$$X_2^{\bar{c}} = \begin{cases} 0, & \text{if event C occurs,} \\ X_2, & \text{if event C it does not occur,} \end{cases}$$

we obtain:

$$M[X_1/\bar{C}] = \frac{1}{P(\bar{C})} M[X_1^{\bar{c}}], \tag{12.10}$$

$$M[X_2/C] = \frac{1}{P(C)} M[X_2^{c}], \tag{12.11}$$

$$M[X_2/\bar{C}] = \frac{1}{P(\bar{C})} M[X_2^{\bar{c}}]. \tag{12.12}$$

Let us note that for any moment of time t

$$\left.\begin{aligned} X_1^c + X_1^{\bar{c}} = X_1; \quad X_2^c + X_2^{\bar{c}} = X_2; \\ X_1^c + X_1^{\bar{c}} + X_2^c + X_2^{\bar{c}} = N. \end{aligned}\right\} \tag{12.13}$$

Now let us pass to the derivation of differential equations for describing the process, which takes place in our system. In this case, we will proceed from the fact that the numbers of states in the case when stabilizer works, are approximately equal to the conditional mathematical expectations of these numbers when occurs

event C; but when it it does not work - to the corresponding conditional mathematical expectations when occurs event $\bar{C}$.

First of all, let us describe our system with the help of graph. This graph/count (Fig. (6.41) will appear somewhat differently in comparison with the usual case. He falls into two subgraph. First (upper) - this of the subgraphs of the states of the stabilizer which can be in one of the two states:

C - works,

$\bar{C}$ - does not work (it is overhauled).

As far as instrument is concerned, for it we is considered to be located in one of the four states:

Π_1^c - instrument works with working stabilizer,

Π_2^c - an instrument does not work (it is overhauled) with working stabilizer,

$\Pi_1^{\bar{c}}$ - an instrument works with the inoperative stabilizer,

$\Pi_2^{\bar{c}}$ - an instrument is overhauled with the inoperative stabilizer.

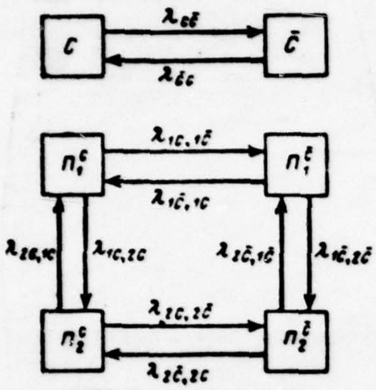

Fig. 6.41.

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the state of stabilizer at torque/moment t is characterized by one of from existences $C = C(t)$, $\bar{C} = \bar{C}(t)$ (until now, we for brevity always lowered t). Of the probabilities of these events let us designate p(t) and $\bar{p}(t) = 1 - p(t)$. As is evident, these are the already known to us probabilities of the states of stabilizer.

Of the numbers of states $\Pi_1^c, \Pi_2^c, \Pi_1^{\bar{c}}, \Pi_2^{\bar{c}}$ we already introduced into the examination: this $X_1^c = X_1^c(t)$, $X_2^c = X_2^c(t)$, $X_1^{\bar{c}} = X_1^{\bar{c}}(t)$, $X_2^{\bar{c}} = X_2^{\bar{c}}(t)$.

Corresponding mathematical expectations let us designate:

$$\left.\begin{aligned} m_1^c(t) &= M[X_1^c(t)], \\ m_2^c(t) &= M[X_2^c(t)], \\ m_1^{\bar{c}}(t) &= M[X_1^{\bar{c}}(t)], \\ m_2^{\bar{c}}(t) &= M[X_2^{\bar{c}}(t)]. \end{aligned}\right\} \quad (12.14)$$

It is obvious, for any moment of time t

$$m_1^c(t) + m_2^c(t) + m_1^{\bar{c}}(t) + m_2^{\bar{c}}(t) = N. \quad (12.15)$$

Let us determine the intensities of flow of events for the graph of Fig. 6.41. First of all, according to condition;

$$\lambda_{c\bar{c}} = \lambda^c = \varphi(X_1), \quad (12.16)$$

$$\lambda_{\bar{c}c} = \frac{1}{\bar{t}^c_{\text{рем}}} = \frac{1}{\psi(X_2)}. \quad (12.17)$$

Further, instrument passes from state Π_1^c in $\Pi_1^{\bar{c}}$ or from state Π_2^c into $\Pi_2^{\bar{c}}$ not by itself, but only together and it is simultaneous with the stabilizer (when that goes out of order); therefore

$$\lambda_{1c,1\bar{c}} = \lambda_{2c,2\bar{c}} = \lambda_{c\bar{c}} = \varphi(X_1). \quad (12.18)$$

It is analogous,

$$\lambda_{1\bar{c},1c} = \lambda_{2\bar{c},2c} = \lambda_{\bar{c}c} = \frac{1}{\psi(X_2)}. \quad (12.19)$$

As concerns the transitions of instrument from Π_1^c, $\Pi_1^{\bar{c}}$ in Π_2^c, $\Pi_2^{\bar{c}}$ and vice versa (on vertical arrow/pointers), it is not difficult to establish/install the appropriate intensities:

$$\lambda_{1c,2c} = \mu_c, \quad \lambda_{1\bar{c},2\bar{c}} = \mu_{\bar{c}}; \quad (12.20)$$

$$\lambda_{2c,1c} = \frac{1}{t_c(X_3)}, \quad (12.21)$$

$$\lambda_{2\bar{c},1\bar{c}} = \frac{1}{t_{\bar{c}}(X_3)}. \quad (12.22)$$

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Now, according to our modification of the principle of quasi-regularity, during the composition of differential equations we must replace X_1, X_2 by their conditional mathematical expectations; namely, where occurs speech about transitions from the left side of the graph/count (into left or into right) - by conditional mathematical expectations when the stabilizer is exact (condition C); and where the transitions are accomplished from right side - under condition $\bar{C}$. This means that in formulas (12.16), (12.18), (12.21) we will replace

$$X_1 \overset{\text{on}}{\cancel{\text{на}}} M[X_1/C],\ X_2 \overset{\text{on}}{\cancel{\text{на}}} M[X_2/C],$$

a in formulas (12.17), (12.19), (12.22)

$$X_1 \overset{\text{on}}{\cancel{\text{на}}} M[X_1/\bar{C}],\ X_2 \overset{\text{on}}{\cancel{\text{на}}} M[X_2/\bar{C}].$$

Since formulas (12.20) do not contain X_1, X_2, then in them nothing to substitute not is necessary.

Using formulas (12.9)-(12.12), we find the conditional mathematical expectations:

$$\left.\begin{aligned} M[X_1/C] &= \frac{1}{p(t)} m_1^C(t), \\ M[X_2/C] &= \frac{1}{p(t)} m_2^C(t), \\ M[X_1/\bar{C}] &= \frac{1}{\bar{p}(t)} m_1^{\bar{C}}(t), \\ M[X_2/\bar{C}] &= \frac{1}{\bar{p}(t)} m_2^{\bar{C}}(t). \end{aligned}\right| \quad (12.23)$$

Thus, we can finally write out mixed type differential equations, which approximately describe our system (argument t for brevity let us lower):

$$
\left.
\begin{aligned}
&\frac{dp}{dt} = -\varphi\left(\frac{m_1^c}{p}\right)\cdot p + \frac{1}{\psi\left(\frac{m_2^{\bar c}}{\bar p}\right)}\cdot \bar p;\\
&\frac{d\bar p}{dt} = -\frac{1}{\psi\left(\frac{m_2^{\bar c}}{\bar p}\right)}\cdot \bar p + \varphi\left(\frac{m_1^c}{p}\right)\cdot p;\\
&\frac{dm_1^c}{dt} = -\varphi\left(\frac{m_1^c}{\bar p}\right)\cdot m_1^c + \mu_c\cdot m_1^c +\\
&\qquad + \frac{1}{\psi\left(\frac{m_2^{\bar c}}{\bar p}\right)}\cdot m_1^{\bar c} + \frac{1}{l_c\left(\frac{m_2^c}{p}\right)}\cdot m_2^c;\\
&\frac{dm_2^c}{dt} = -\varphi\left(\frac{m_1^c}{p}\right)\cdot m_2^c - \frac{1}{l_c\left(\frac{m_2^c}{p}\right)}\cdot m_2^c +\\
&\qquad + \mu_c\, m_1^c + \frac{1}{\psi\left(\frac{m_2^{\bar c}}{\bar p}\right)}\cdot m_2^{\bar c};\\
&\frac{dm_1^{\bar c}}{dt} = -\frac{1}{\psi\left(\frac{m_2^{\bar c}}{\bar p}\right)}\cdot m_1^{\bar c} - \mu_c\, m_1^{\bar c} +\\
&\qquad + \varphi\left(\frac{m_1^c}{p}\right)\cdot m_1^c + \frac{1}{l_{\bar c}\left(\frac{m_2^{\bar c}}{\bar p}\right)}\cdot m_2^{\bar c};\\
&\frac{dm_2^{\bar c}}{dt} = -\frac{1}{\psi\left(\frac{m_2^{\bar c}}{\bar p}\right)}\cdot m_2^{\bar c} - \frac{1}{l_{\bar c}\left(\frac{m_2^{\bar c}}{\bar p}\right)}\cdot m_2^{\bar c} +\\
&\qquad + \mu_{\bar c}\, m_1^{\bar c} + \varphi\left(\frac{m_1^c}{p}\right)\cdot m_2^c.
\end{aligned}
\right\}
\qquad (12.24)
$$

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Let us note that from this system of equations it is possible to

exclude two equations: one of first twc, using condition $p + \bar{p} = 1$, and one - of subsequent four, using relationship/ratio (12.15).

These equations can be solved under any initial conditions; for example, if in the beginning stabilizer and all instruments work:

$$t=0;\quad p=1;\quad \bar{p}=0;\quad m_1^c=N;$$
$$m_2^c=m_1^{\bar{c}}=m_2^{\bar{c}}=0.$$

If to us it is important to study, let us say, how rapidly system leaves the "block", created by random breakdown of the considerable number of instruments (L) and stabilizer, initial conditions must be selected by others:

$$t=0;\quad p=0;\quad \bar{p}=1;\quad m_2^c=0;\quad m_2^c=0;$$
$$m_1^{\bar{c}}=N-L;\quad m_2^{\bar{c}}=L.$$

13. Some refinements of the method of the dynamics of average.

Until now, examining the equations of the dynamics of average, we everywhere used the principle of quasi-regularity. Recall that of what consists this principle. If the intensities of flow of the events, translating the elements of system of one state into another, in a specific manner depended on the numbers of states, we substituted in the expressions of these dependences numbers themselves (random) by their average values - mathematical expectations.

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The same, although somewhat a complicated form, we made in mixed type equations substituting the arguments on which depended the intensities, by conditional mathematical expectations. In this case, the accuracy and the acceptability of the very principle of quasi-regularity by us was not considered.

In actuality principle itself represents by itself certain assumption, and with the use by it we unavoidably allow/assume some errors. We already mentioned about the fact that these errors are comparatively small for the cases when the number of cell/elements in system is great, and are not also small the average numbers of those states on which depend the intensities. In this paragraph we will touch a question concerning the errors inherent in the method of the dynamics of average, connected with the principle of quasi-regularity and will introduce into the equations of the dynamics of average some refinements which will allow, in the first approximation, to evaluate the order of these errors.

For simplicity we will consider the case when a cell/element ε have a total of two state: ε_1 and ε_2, and from number X_1 of state ε_1 depends only one intensity λ_{12}, but intensity λ_{21} is constant: λ_{21} = const. The graph/count of the states of cell/element

$\mathcal{E}$ is given in Fig. [redacted] 6.42.

For future reference to us it is convenient it will be to introduce special designation $\Lambda_{12}(X_1)$ for the total intensity of flow of the events, which translate the elements of system from state $\mathcal{E}_1$ into $\mathcal{E}_2$, a the intensity λ_{12} of the flow, which functions on one cell/element, to express by this total intensity:

$$\lambda_{12}=\frac{\Lambda_{12}(X_1)}{X_1}. \tag{13.1}$$

It turns out that for the average numbers $m_1(t)$, $m_2(t)$ states $\mathcal{E}_1$, $\mathcal{E}_2$ it is possible to deduce, without using the principle of quasi-regularity, the completely precise differential equations, which express the derivatives dm_1/dt, dm_2/dt through the mathematical expectation of random variable $\Lambda_{12}(X_1)$. A namely:

$$\frac{dm_1}{dt}=-M[\Lambda_{12}(X_1)]+\lambda_{21}m_2 \tag{13.2}$$

(equation for m_2 we do not write out, since in this case, $\frac{dm_2}{dt}=\frac{d(N-m_1)}{dt}=-\frac{dm_1}{dt}$).

Let us show how is derive/concluded equation (13.2). For this, let us consider the graph/count of the states no longer of one separate cell/element, but systems as a whole (Fig. 6.43). The states of system $S_0, S_1, S_2, \ldots, S_N$ let us label correspondingly to number X_1 of cell/elements, which are found in state $\mathcal{E}_1$.

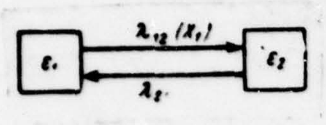

Fig. 6.42.

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With the large number of cell/elements N, the number of states is extremely great, and composition and the solution of the system of differential equations for the probabilities of the states of system difficultly; for this very reason turn we to the method of the dynamics of average. All the same we will register the equations of Kolmogorov for the probabilities of the states of system S (since we are not gathered to solve them, the number of equations to us it is unimportant). System of equations takes the form:

$$\left.\begin{aligned} &\frac{dp_0}{dt} = -N\lambda_{21}p_0 + \Lambda_{12}(1)p_1; \\ &\cdots\cdots\cdots\cdots \\ &\frac{dp_k}{dt} = -(N-k)\lambda_{21}p_k - \Lambda_{12}(k)p_k + \\ &\quad + (N-k+1)\lambda_{21}p_{k-1} + \Lambda_{12}(k+1)p_{k+1}; \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &\frac{dp_N}{d} = -\Lambda_{12}(N)p_N + \lambda_{21}p_{N-1}, \end{aligned}\right\} \tag{13.3}$$

where $p_k(t) = P(X_1 = k)$ - probability that at torque/moment t the system will be able S_k (k = 0, 1, 2, ..., N).

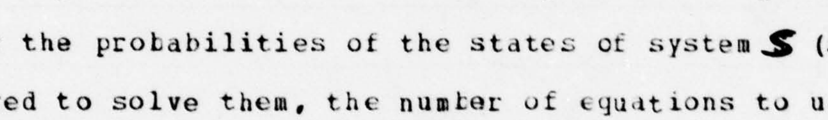

Fig. 6.43.

Let us note that the first and latter of equation (13.3) can be reduced to the general view, in which is registered $\frac{dp_k}{dt}$, if we naturally assume

$$\Lambda_{12}(0)=0; \quad p_{-1}=p_{N+1}=0. \tag{13.4}$$

We know that the mathematical expectation of discrete random variable $X_1(t)$ possible values of which - integers from 0 to N, is expressed by the formula:

$$m_1(t)=M[X_1(t)]=\sum_{k=0}^{N} k\,p_k(t). \tag{13.5}$$

Therefore derivative of this mathematical expectation we will obtain, after multiplying the k equation of system (13.3) by k it summed up from 0 to N:

$$\frac{dm_1}{dt}=-\sum_{k=0}^{N} k(N-k)\lambda_{21}p_k-\sum_{k=0}^{N} k\Lambda_{12}(k)p_k+$$
$$+\sum_{k=0}^{N} k(N-k+1)\lambda_{21}p_{k-1}+\sum_{k=0}^{N} k\Lambda_{12}(k+1)p_{k+1}. \tag{13.6}$$

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The first two sums of this expression will leave as they there is, and the third and fourth we convert. Let us consider the third sum. Taking into account that in this sum the term, which corresponds to k = 0, is converted into zero, we have:

$$\sum_{k=0}^{N} k(N-k+1)\lambda_{21}p_{k-1}=\sum_{k=1}^{N} k(N-k+1)\lambda_{21}p_{k-1}. \qquad (13.7)$$

Let us further change the index of addition, after placing k-1 = i:

$$\sum_{k=1}^{N} k(N-k+1)\lambda_{21}p_{k-1}=\sum_{i=0}^{N-1}(i+1)(N-i)\lambda_{21}p_i=$$
$$=\sum_{i=0}^{N}(i+1)(N-i)\lambda_{21}p_i. \qquad (13.8)$$

Last/latter equality is correct, since with i = N factor (not) is converted into zero. Finally, being returned to designation k for the index of the addition (recall that the sum does not depend on by which letter to designate this index), we will obtain expression for the third sum:

$$\sum_{k=0}^{N} k(N-k+1)\lambda_{21}p_{k-1}=\sum_{k=0}^{N}(k+1)(N-k)\lambda_{21}p_k. \qquad (13.9)$$

We analogously convert fourth sum; taking into account that $p_{N+1}=\Lambda_{12}(0)=0$, we have:

$$\sum_{k=0}^{N} k\Lambda_{12}(k+1)\,p_{k+1}=\sum_{k=0}^{N-1} k\Lambda_{12}(k+1)\,p_{k+1}=$$
$$=\sum_{i=1}^{N}(i-1)\,\Lambda_{12}(i)\,p_i=\sum_{i=0}^{N}(i-1)\,\Lambda_{12}(i)\,p_i=$$
$$=\sum_{k=0}^{N}(k-1)\,\Lambda_{12}(k)\,p_k. \qquad (13.10)$$

Let us substitute expressions (13.9) and (13.10) into formula (13.6):

$$\frac{dm_1}{dt}=-\sum_{k=0}^{N}k(N-k)\,\lambda_{21}\,p_k+\sum_{k=0}^{N}k\Lambda_{12}(k)\,p_k+$$
$$+\sum_{k=0}^{N}(k+1)(N-k)\,\lambda_{21}\,p_k+\sum_{k=0}^{N}(k-1)\,\Lambda_{12}(k)\,p_k=$$
$$=\lambda_{21}\sum_{k=0}^{N}(N-k)\,p_k-\sum_{k=0}^{N}\Lambda_{12}(k)\,p_k. \qquad (13.11)$$

Here the first sum - nothing else but $M[X_2]$, i.e., m_2, but the second - this $M[\Lambda_{12}(X_1)]$. Thus, we deduced average number m_1 of state $\mathcal{E}_1$.

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However, this equation in its precise form for us is completely useless. The fact is that in its right side enter only unknown functions m_1 and m_2, but also mathematical expectation $M[\Lambda_{12}(X_1)]$. But in order to know this mathematical expectation, we should know the large

number (N) of probabilities $p_k(t)$ $(k = 1, \ldots, N)$. Of course, it is possible in principle to find them, solving system (13.3) ; but we for that we apply the method of the dynamics of average, in order to avoid the solution of the large number of equations for the probabilities of the states of system.

Arises the question concerning how to find approximately mathematical expectation $M[\Lambda_{12}(X_1)]$, without knowing the probabilities of the states of system p_k (K = 1, 2, ..., N).

One Of the methods, which make it possible to find approximate value $M[\Lambda_{12}(X_1)]$ - this is the principle of quasi-regularity which we, until now, used. It consists actually of the fact that we approximately replace the mathematical expectation of function from random variable with the same function from mathematical expectation, i.e., we set/assume:

$$M[\Lambda_{12}(X_1)] \approx \Lambda_{12}(M[X_1]) = \Lambda_{12}(m_1). \tag{13.12}$$

After this precise equation (13.2) is converted into the approximate equation

$$\frac{dm_1}{dt} \approx -\Lambda_{12}(m_1) + \lambda_{21} m_2,$$

or, if we use intensity in recalculation to one cell/element:

$$\frac{dm_1}{dt} \approx -\lambda_{12}(m_1)\, m_1 + \lambda_{21} m_2.$$

Thus, error during the application/use of principle of quasi-regularity - the same as error from the replacement of the mathematical expectation of function by the same function from mathematical expectation.

Relative to the error, which appears during this replacement, it is possible to express following overall considerations. this error is small, if function $\Lambda_{12}(x)$ is almost linear in the range of the virtually possible values of random variable X_1. If in this range function $\Lambda_{12}(x)$ strongly differs from the linear, error can be considerable. If function $\Lambda_{12}(x)$ is convex upward, as this is typical for the tasks of the dynamics of average (Fig. 6.44), then the error from the application/use of formula (13.12) will be always to large side, i.e.,

$$\Lambda_{12}(m_1) > M[\Lambda_{12}(X_1)].$$

For the function $\Lambda_{12}(x)$, convex downward, the error, on the contrary, will be to smaller side. However, these considerations do not make it possible to consider the value of error.

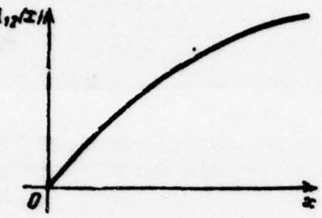

Fig. 6.44.

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So that at least it is rough to consider the error in approximation formula (13.12), it is possible to use the following method. We know that if the intensities of flow of the events, which translate cell/elements from state into state, do not depend on very numbers of states (i.e. cell/elements pass from state into the state independent from each other), then the numbers of states will be distributed according to the binomial law (see §1). In particular, the number of state $\mathcal{E}_1$ will be distributed according to binomial law with the mathematical expectation m_1 and root-mean-square deviation $\sigma_1 = \sqrt{m_1\left(1 - \frac{m_1}{N}\right)}$, where N - total number of cell/elements in system. We know also (see 2) that if the intensities of flow of events depend on the numbers of states, then this, generally speaking, not then. However, for the roughly approximate account to the chance of argument X_1 in function $\Lambda_{12}(X_1)$ let us assume that and in this case

the law of the distribution of the number of state will be binomial, with the mathematical expectation m_1 and root-mean-square deviation $\sigma_1 = \sqrt{m_1\left(1-\frac{m_1}{N}\right)}$. This, of course, will be inaccurate, but all the same it is much more precise than it is simple to set/assume number X_1 not random and equal to its mathematical expectation (that we actually make, using the principle of quasi-regularity).

Let us register this probability distribution. Probability that the number of state ε_1 will be equal k, is expressed by the known formula:

$$p_k = C_N^k \left(\frac{m_1}{N}\right)^k \left(1-\frac{m_1}{N}\right)^{N-k}. \qquad (13.13)$$

Thus (if we consider that X_1 has the binomial distribution) $M[\Lambda_{12}(X_1)]$ it will be expressed by the formula:

$$M[\Lambda_{12}(X_1)] = \sum_{k=0}^{N} \Lambda_{12}(k)\, C_N^k \left(\frac{m_1}{N}\right)^k \left(1-\frac{m_1}{N}\right)^{N-k}. \qquad (13.14)$$

With the large number of cell/elements of calculation on formula (13.14) are very bulky; in order to avoid this, it is possible to use the maximum properties of binomial distribution with the large number of experiments. It is known that the binomial distribution with the large number of experiments N under some conditions approaches normal, and in others - the Poisson distribution (see for example [7]). The first case will occur when the probability of event in each experiment is not too small and too great; this, can be judged by the

fact that entire interval $m_1 \pm 3\sigma_1$ is placed on section (0, N), i.e.,

$$m_1 - 3\sigma_1 > 0; \quad m_1 + 3\sigma_1 < N. \tag{13.15}$$

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If both these conditions are satisfied, then the average value of intensity $M[\Lambda_{12}(X_1)]$ can be computed, approximately substituting discrete random variable X_1 continuous, distributed according to normal law, and sum (13.14) - by the integral:

$$M[\Lambda_{12}(x_1)] = \int_0^N \Lambda_{12}(x) f(x)\, dx, \tag{13.16}$$

where

$$f(x) = \frac{1}{\sigma_1\sqrt{2\pi}} e^{-\frac{(x-m_1)^2}{2\sigma_1^2}}. \tag{13.17}$$

Condition (13.15) with large N can not be implemented in two cases.

1. When average number M_1 of cell/elements in state $\mathcal{E}_1$ is too small in comparison with N; then

$$\sigma_1^2 = m_1\left(1 - \frac{m_1}{N}\right) \approx m_1, \tag{13.18}$$

i.e. dispersion of value X_1 is approximately equal to its mathematical expectation, and this - sign/criterion of fact that binomial distribution is close to Poisson.

2. When average number m_1 of cell/elements in state $\mathcal{E}_1$, on the contrary, is close to N and, which means, that according to Poisson law is distributed not X_1, but its addition to N, i.e., random value $Y_1 = N-X_1$ [1].

FOOTNOTE [1]. V our case $Y_1 = X_2$, but if the number of states is greater than two, this will be no longer then; therefore we will retain for random variable a $N-X_1$ separate designation Y_1. ENDFOOTNOTE.

Let us show as to compute approximately value $M[\Lambda_{12}(X_1)]$ in both cases.

1. Random variable X_1 is distributed according to the law of Poisson with mathematical expectation m_1. The mathematical expectation of its function $\Lambda_{12}(X_1)$ is equal

$$M[\Lambda_{12}(X_1)] = \sum_{k=0}^{N} \Lambda_{12}(k)\, p_k, \qquad (13.19)$$

where

$$p_k = \frac{m_1^k}{k!} e^{-m_1}.$$

For calculations according to formula (13.19) can be used the

tables of the Poisson distribution (delays from tables gives, for example, in appendix, table 2).

2. The random value $Y_1 = N-x_1$ is distributed according to the law of Poisson with the mathematical expectation $N-m_1$. The mathematical expectation of function $\Lambda_{12}(X_1)$ will be expressed by the formula

$$M[\Lambda_{12}(X_1)] = \sum_{k=0}^{N} \Lambda_{12}(N-k) p_k^*, \qquad (13.20)$$

where

$$p_k^* = \frac{(N-m_1)^k}{k!} e^{-(N-m_1)}$$

- probabilities of the Poisson distribution, also determined on tables.

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Let us assume that we approximately expressed thus $M[\Lambda_{12}(X_1)]$ in the form of certain function $\bar{\Lambda}_{12}(m_1)$; this function will be assigned by three different formulas (13.16), (13.19) and (13.20) depending on that, on which part of the segment (0, N) it is located m_1. It is certain, it would be possible to substitute the appropriate expression into equation (13.2) for the average number m_1 (in this case it suffices to solve this alone equation):

$$\frac{dm_1}{dt} = -\bar{\Lambda}_{12}(m_1) + \lambda_{21} m_2, \qquad (13.21)$$

but it will render/show too complex. Therefore a task has sense approximately to solve in two stages. First (in the first approximation,) to solve the equations of the dynamics of average, obtained with the help of the usual principle of quasi-regularity. Then, after considering in the first rough approximation the average number of state $\mathcal{S}_1$ - value m_1 - to find approximate value for

$$M[\Lambda_{12}(X_1)] \approx \bar{\Lambda}_{12}(m_1) = \Lambda_{12}^*(t)$$

(without fail for a series of the values of value t), using in this case that or other of formulas (13.16), (13.19), (13.20). Between the obtained thus values $\Lambda_{12}^*(t)$ it is possible to interpolate intermediate. Thus is constructed the function of time $\Lambda_{12}^*(t)$, which is substituted in the right side of equation (13.2):

$$\frac{dm_1}{dt} = -\Lambda_{12}^*(t) + \lambda_{21}(N - m_1). \qquad (13.22)$$

Is obtained the linear differential equation with variable coefficients whose solution difficulties is not caused. As a result of this, will be obtained function $m_1(t)$, more precise than the first approximation. Equate/comparing the second approach/approximation with the first, it is possible to approximately consider the errors, which appear from the application/use of principle of quasi-regularity.

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In perfect analogy it is possible to solve the task of refining the equations and then, when the number of states of cell/element is more than two and when on the numbers of states depends not one intensity, but two or it is more. Entire/all difference in the fact that it is necessary to consider the mathematical expectation not of one function, but several.

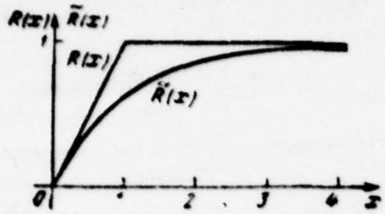

Fig. 6.45.

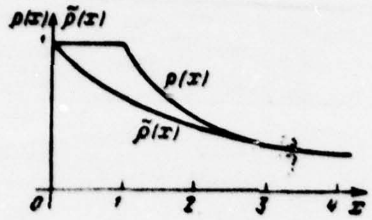

Fig. 6.46.

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Method presented above of the introduction of corrections to equations of the dynamics of average is comparatively laborious; however for the functions of the total intensity of some special forms, which are frequently encountered in the equations of the dynamics of average, correction they can be taken into account sufficiently simply.

For example, under conditions of the simplest task with the graph/count of the states of the cell/element (see Fig. 6.42) total intensity $\Lambda_{12}(X_1)$ it is equal to constant λ_0 at all values $X_1 = 1, 2, \ldots$, (but with $X_1 = 0$, is logical, $\Lambda_{12}(X_1) = 0$). Then, if m_1 is great, then $M[\Lambda_{12}(X_1)] \approx \lambda_0$ with very high accuracy. In order to approximately find this mathematical expectation at the small values m_1, let us take for value X_1 Poisson distribution with parameter m_1. Then we

obtain:

$$M[\Lambda_{12}(X_1)] = 0\,p_0 + \lambda_0 p_1 + \lambda_0 p_2 + \ldots + \lambda_0 p_N =$$
$$= \lambda_0 (p_1 + p_2 + \ldots + p_N) = \lambda_0 (1 - p_0) \approx \lambda_0 [1 - e^{-m_1}]. \quad (13.23)$$

Let us designate function $1-e^{-x}$ through $\tilde{R}(x)$, and $\frac{\tilde{R}(x)}{x}$ through $\tilde{\rho}(x)$. The approximate equation for the average number m_1 will be registered then thus:

$$\left.\begin{aligned} \frac{dm_1}{dt} &= -\lambda_0 \tilde{R}(m_1) + \lambda_{21} m_2, \\ \text{or} \quad \frac{dm_1}{dt} &= -\lambda_0 \tilde{\rho}(m_1) m_1 + \lambda_{21} m_2. \end{aligned}\right\} \quad (13.24)$$

Let us note that the less precise equation, obtained for the principle of quasi-regularity, would here take the form:

$$\left.\begin{aligned} \frac{dm_1}{dt} &= -\lambda_0 R(m_1) + \lambda_{21} m_2, \\ \text{or} \quad \frac{dm_1}{dt} &= -\lambda_0 \rho(m_1) m_1 + \lambda_{21} m_2, \end{aligned}\right\} \quad (13.25)$$

where R(x) and ρ(x) of functions, introduced in §4.

Is represented, for a comparison, plotted functions R(x) and $\tilde{R}$(x) (Fig. 6.45) and functions ρ(x) and $\tilde{\rho}$(x) (Fig. 6.46).

As can be seen from graphs, the error during the replacement of right side in equations (13.24) by the corresponding right side in equations (13.25) is sufficiently essential in the small values m_1.

whereas with the large m_1 it becomes negligible.

Thus, in all tasks where we used functions R, ρ as correction factors in the right sides of the equations of the dynamics of average, more accurate results they will be obtained, if we replace R for $\tilde{R}$, ρ on $\tilde{\rho}$.

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7. METHODS OF ACCOUNT TO THE RELIABILITY OF TECHNICAL EQUIPMENT

1. Estimation problem of reliability.

The overwhelming majority of the operations, which are subject to quantitative research, in contemporary society is implemented with the application/use of one or the other technical equipment/devices. The evaluation of the efficiency of such operations and the consumption/production/generation of rational solutions by their organization require the account to the reliability of technical equipment/devices used.

Hearth by "reliability" in the broad sense is understood the capability of technical equipment/device for a trouble-free (reliable) operation during the assigned time interval under certain conditions. This time interval is usually caused by the time of the execution of certain task which is realized by technical equipment/device and is the part of the common/general/total task of operation.

At present, in connection with the increasing complexity of technical equipment/devices and the widespread introduction of automation in all the regions of practice, the problem of reliability becomes one it becomes one of the junction/unit problems of technology and organization of control. Provision for a reliable work of all equipment components - task of paramount importance.

Fight for reliability requires special examination and the quantitative analysis of the phenomena, connected with the chance failures of equipment. In recent years the theory of reliability was converted into the special science, using extensively the probabilistic methods of study.

In the theory of reliability, is accepted to distinguish two types of the failures: sudden and gradual.

Under the random failure of equipment/device, is understood the instantaneous breakdown, which indicates the impossibility of its application/use. The random failure appears in some, generally speaking, the random moment of time. By examples of the random failures can serve: the burnout electrc or of electron tube, the break of conductor, as the sample/test of condenser/capacitor, etc.

Under gradual is understood the failure of equipment/device, connected with gradual deterioration ("creeping") in its characteristics. For the elimination of such failures, is required the control of instrument.

The deterioration failures can be conditionally considered as sudden, if we agree to consider that some deviations of the parameters of equipment/device from rating they are still permissible, and large - not admitted; as soon as the parameters they exceed these limits, equipment/device is considered refused.

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However, the destination of such limits in a series of the cases is difficult. Will more right consider the parameters of equipment/device as random functions of time, connect with them some index of the efficiency of equipment/device (for example, the probability of the solution of problem or the mathematical expectation of productivity) and this index compute taking into account the "creeping" of characteristics. This approach requires the attentive study of structure and work of concrete/specific/actual technical equipment/device and application/use of a comparatively

complex mathematical apparatus. In this chapter we will examine only random failures.

The reliability of technical equipment/device or, seemingly let us speak, system it depends on a composition and a quantity of which form system cell/elements (units), on the method of their association into system and on the characteristics of each separate cell/element.

The division of technical equipment/devices into "systems" and their forming "elements" bears conditional character and depends on the setting of task and target/purposes of research. one and the same equipment/device, for example the radar sight of destroyer, can be considered and as "the system", which consists of the cell/elements: electron tubes, capacitors, relay and so forth, and as "element" of more complex system - equipment of aircraft. in turn, fighter it is the "element" of air defense system

Subsequently we will call "cell/element" any technical equipment/device, which is not subject to further separation whose reliability is considered given one or it is determined experimentally. Combining such cell/elements differently into "systems", we will solve the problem of determining the reliability of system from the reliability of its cell/elements.

2.

Reliability of cell/element. Density of distribution of the time of failure-free operation. Mean time of failure-free operation.

The estimation of the reliability of system and cell/elements requires the introduction of quantitative characteristics. Let us consider here some of these characteristics. For brevity let us determine them in connection with "cell/element"; however the same determinations will be related also to "system".

The reliability of cell/element (in the narrow sense of the word) is called probability that this cell/element under given conditions will work smoothly for a period of time t. This probability we will designate p(t). Function p(t) is called sometimes the "law of reliability".

It is logical, with an increase in the time, function p(t) decreases (Fig. 7.1). With t = 0, is logical to assume p(t) = 1.

The unreliability of cell/element is called probability q(t) of the fact that cell/element will refuse (it will leave the system) for a period of time t. It is obvious,

$$q(t) = 1 - p(t). \qquad (2.1)$$

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Let us consider time T of the failure-free operation of cell/element as random variable. The function of distribution F(t) of this random variable is defined as

$$F(t)=P(T<t). \tag{2.2}$$

Is obvious, F(t) - probability that for time t cell/element will refuse - it represents by itself nothing else but the unreliability of the cell/element:

$$F(t)=q(t), \tag{2.3}$$

a its reliability supplements F(t) to the unit:

$$p(t)=1-F(t). \tag{2.4}$$

Thus, unreliability q(t) possesses the properties of the distribution function of nonnegative random variable. It is equal to zero with t = 0, it does not decrease with increase t and it approaches unity when $t\to\infty$ (Fig. 7.2).

In practice usually instead of the function of distribution F(t) they use it derivative - density of distribution or probability

density:

$$f(t) = F'(t) = q'(t). \quad (2.5)$$

The graph of density f(t) is shown on Fig. to 7.3. The area, limited curved f(t), is equal to unit.

Value f(t)dt - probability element - is construed as probability that time T will take the value, which lies within the limits of elementary section (t, t + dt).

In the literature on reliability function f(t) frequently calls the "density of failures". To avoid the misunderstandings, connected with ill-defined terminology, we will call f(t) it is more accurately: the density of distribution of the time of failure-free operation.

Density f(t) can be approximately determined from experiment, for which is placed the following experiment: is observed the work of the large number N of uniform cell/elements; each of them works to the torque/moment of failure. The time, during which worked the cell/element, is recorded. The obtained values of the time:

$$t_1, t_2, \ldots, t_N$$

are treated by the usual methods of the mathematical statistics: is constructed histogram (Fig. 7.4) and is equalized with the help of any smooth curved, which possesses properties density.

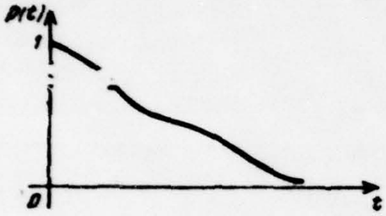

Fig. 7.1.

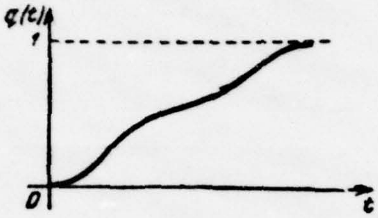

Fig. 7.2.

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The ordinate of histogram on each elementary section of time Δt represents by itself nothing else but the average number of failures for time unit, which is necessary to one tested cell/element. The same sense can be ascribed and to function f(t). Approximately density f(t) is determined from the formula

$$f(t) \approx \frac{m(t,\ t+\Delta t)}{N\,\Delta t}. \qquad (2.6)$$

where m(t, t + Δt) - a number of cell/elements, which refused on the section of time from t to t + Δt (time is counted off from the torque/moment of connection/inclusion); N - total number of cell/elements; Δt - length of the elementary section of time.

Example. Was tested N = 1000 tubes to the duration of

failure-free operation. Test results are given in Table 2.1.

To find approximately density f(t) for each section of time, to construct histogram and to straighten (by hand) smooth curve.

Solution. On the first section (0-10 hour) we have:

$$f(t) \approx \frac{151}{1000 \cdot 10} \approx 0{,}0151,$$

on the second

$$f(t) \approx \frac{102}{1000 \cdot 10} \approx 0{,}0102$$

and so forth. The values of density f(t) are given in Table 2.2.

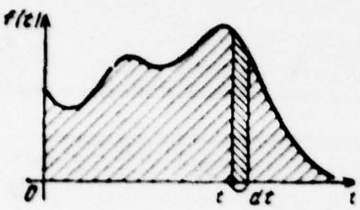

Fig. 7.3.

Fig. 7.4.

Table 2.1.

Длительность работы в часах (от—до) (1)	0—10	10—20	20—30	30—40	40—50	50—60	60—80	80—100	100—150	150—200
Число ламп m $(t, t+\Delta t)$ (2)	151	102	77	81	79	120	200	69	91	50

Key: (1). Duration of work in hours (from - to). (2). Number of tubes.

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Histogram and the leveling curve are given to Fig. by 7.5.

Let us note that density f(t), depicted in Fig. 7.5, has a maximum with t = 0, i.e., maximum failure rate falls on the initial operating cycle of cell/element. This character curved f(t) frequently is observed in practice, especially in work from electro- and by radio parts, as they frequently have a tendency to reject

immediately or soon after connection/inclusion. Sometimes this increase of density at point $t = 0$ manifests itself so sharply that the noticeable fraction of cell/elements can be considered refused accurately at the moment of connection/inclusion. In this case, the time of the failure-free operation T is converted from continuous into mixed random variable in which one the value ($t = 0$) possesses the different from zero probability p_0, and for others there is only some density of distribution. The distribution function of this random variable is shown on Fig. 7.6 - at point $t = 0$ it it has a jump, equal to p_0, and with $t > 0$ it is continuous.

Differentiating function $F(t)$ with $t > 9$, we will obtain the curve of the "density" of $\tilde{f}(t)$ (Fig. 7.7). It is characteristic that that it limits the area, equal no longer to unit, but $1-p_0$. During processing of experimental data in that case, take/select into separate group the cell/elements, which refused during connection/inclusion, and the ratio of their number m_0 to total number N of the tested cell/elements count for approximate values p_0

$$p_0 \approx \frac{m_0}{N},$$

and for remaining data is constructed conventional histogram (in this case frequencies are located by the division of the number of observations in discharge into the total number of observations N).

Table 2.2.

Длительность (1) работы в часах (от—до)	0—10	10—20	20—30	30—40	40—50	50—60	60—80	80—100	100—150	150—200
Плотность (2) $f(t)$	0,0151	0,0102	0,007.	0,0061	0,0079	0,0120	0,0100	0,0038	0,0017	0,0010

Key: (1). Duration of work in hours. (2). Density.

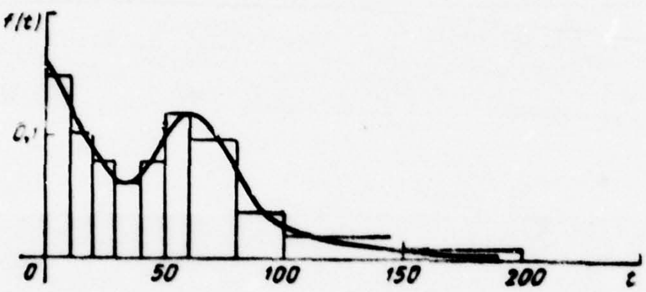

Fig. 7.5.

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As the characteristic of the reliability of cell/element, frequently is applied the mean time of failure-free operation, i.e., the mathematical expectation of value T:

$$\bar{t} = M[T].$$

If value T is continuous (i.e. its function of distribution F(t) does not have gallop with t = 0)

$$\bar{t} = M[T] = \int_0^\infty t f(t)\, dt. \qquad (2.7)$$

In the case when T - mixed random variable, and the separate value t = 0 has probability p_0

$$\bar{t} = M[T] = \int_0^\infty t\bar{f}(t)\,dt. \quad (2.8)$$

Value t can be expressed not through density of distribution f(t), but it is direct through reliability p(t). It is real/actual,

$$\bar{t} = \int_0^\infty tf(t)\,dt = \int_0^\infty tq'(t)\,dt = -\int_0^\infty tp'(t)\,dt.$$

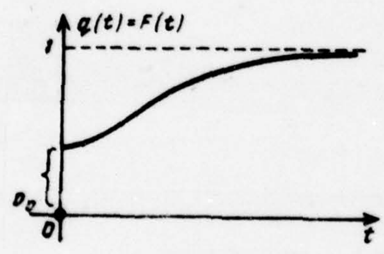

Fig. 7.6.

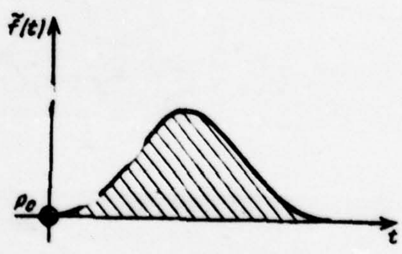

Fig. 7.7.

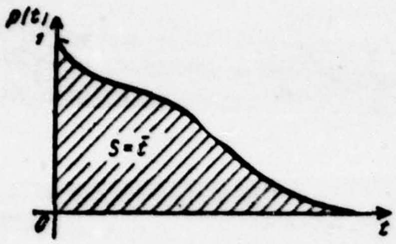

Fig. 7.8.

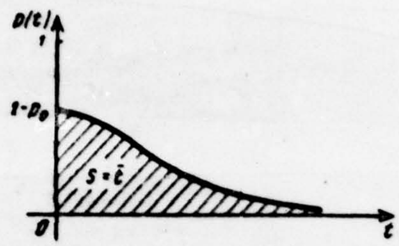

Fig. 7.9.

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Integrating in parts, we have:

$$\bar{t} = -tp(t)\Big|_0^\infty + \int_0^\infty p(t)\,dt. \qquad (2.9)$$

The first member in the right side of expression (2.9) is equal to zero, since for random variable T, in which there is a mathematical expectation, difference 1 - F(t) = p(t) when $t \to \infty$ must decrease faster than increases t. Therefore

$$\bar{t} = \int_0^\infty p(t)\,dt. \qquad (2.10)$$

This formula has the simple geometric interpretation: the mean time of the failure-free operation of cell/element is equal to the full/total/complete area S, limited by reliability curve and by the

axes of coordinates (Fig. 7.8).

It is obvious, in the case when T - mixed random variable (value $t = 0$ has probability p_0), this rule remains valid entire/all difference in the fact that the curve $p(t)$ will begin not from 1, but from $1-p_0$ (Fig. 7.9).

3. Exponential law of reliability. Failure rate.

Most convenient for analytical description is the so-called exponential (or exponential) law of reliability which is expressed by the formula

$$p(t)=e^{-\lambda t}, \qquad (3.1)$$

where $\lambda > 0$ - constant parameter.

The graph of the exponential law of reliability is shown on Fig. 7.10. For this law the function of time allocation of failure-free operation takes the form:

$$F(t)=q(t)=1-e^{-\lambda t}, \qquad (3.2)$$

a density -

$$f(t)=\lambda e^{-\lambda t} \quad (t>0). \qquad (3.3)$$

This be an already known to us exponential law of the distribution according to which is distributed the distance between adjacent events in the simplest flow with intensity λ (see §4 Chapter 4).

In the study of the problems of reliability, frequently there is to convenient visualize the matter so, as if or cell/element functions the simplest flow of failures with intensity λ; cell/element rejects at the torque/moment when comes the first event of this flow.

The form of flow of failures" acquires real sense, if the refused cell/element is immediately substituted new (it is restored).

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The sequence of the random moments of time, at which occur the failures (Fig. 7.11), represents by itself the simplest flow of events, and the intervals between events - the independent random quantities, distributed according to exponential law (3.3).

The concept of "failure rate" can be introduced not only for exponential, but also for any other law of reliability with a density of f(t); entire/all difference will be in the fact that under the

non-exponential law p(t) the rate of failures λ will be no longer constant value, but alternating/variable.

The intensity (or otherwise " danger ") of failures is called relation to the density of distribution of the time of the failure-free operation of cell/element to its reliability:

$$\lambda(t) = \frac{f(t)}{p(t)}. \tag{3.4}$$

Let us explain the physical sense of this characteristic. Let simultaneously test large number N of uniform cell/elements, each - to the torque/moment of their failure. Let us designate n(t) - the number of cell/elements, which render/showed exact up to torque/moment t, but m(t, t + Δt), as before the number of cell/elements, which refused on the low section of time (t, t + Δt). Per time unit, it is necessary the average number of failures

$$\frac{m(t,\ t+\Delta t)}{\Delta t}.$$

Let us divide this value not into the total number of experience/tested cell/elements N, but into number exact up to torque/moment t of cell/elements n(t). It is not difficult to ascertain that with large N this sense will be approximately equal to rate of failures λ(t):

$$\lambda(t) \approx \frac{m(t,\ t+\Delta t)}{n(t)\,\Delta t}. \tag{3.5}$$

It is real/actual, with large N

$$n(t) \approx Np(t)$$

and

$$\frac{m(t, t+\Delta t)}{n(t)\,\Delta t} \approx \frac{m(t, t+\Delta t)}{N\,\Delta t\, p(t)}.$$

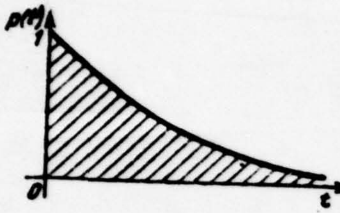

Fig. 7.10.

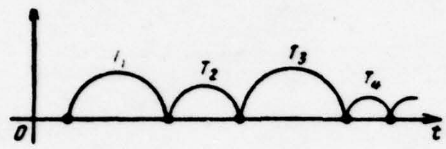

Fig. 7.11.

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But according to formula (2.6)

$$\frac{m(t, t+\Delta t)}{N\,\Delta t} \approx f(t),$$

whence

$$\frac{m(t, t+\Delta t)}{N\,\Delta t p(t)} \approx \frac{f(t)}{p(t)} = \lambda(t).$$

In works on reliability approximation (3.5) frequently is considered as definition of failure rate, i.e., are defined it as average number of failures per unit time, which is necessary to one working cell/element.

To characteristic $\lambda(t)$ it is possible to give still one the interpretation: this be a conditional probability density of the

failure of cell/element at the given instant t, when to torque/moment t it worked smoothly. It is real/actual, let us consider probability element $\lambda(t)dt$ - probability that for time (t, t + dt) cell/element will pass from state "works" in state "it does not work", when to torque/moment t it worked. In fact, the unconditional failure probability of cell/element on section (t, t + dt) is equal f(t)dt. This - the probability of the coincidence of two events:

A - cell/element worked exactly to torque/moment t.

B - cell/element refused on the section of time (t, t + dt). According to product rule of the probabilities:

$$f(t)\,dt = P(AB) = P(A)\,P(B/A).$$

Taking into account that P(A) - r(t), we will obtain:

$$P(B/A) = \frac{f(t)\,dt}{p(t)} = \lambda(t)\,dt;$$

and value $\lambda(t)$ is nothing else but the conditional probability density of transition from state "works" into state "refused" for torque/moment t.

If is known rate of failures $\lambda(t)$, then it is possible to express by it reliability p(t). Taking into account that f(t) = -p'(t), let us register formula (3.4) in the form:

$$\lambda(t) = -\frac{p'(t)}{p(t)} = -[\ln p(t)]'.$$

Integrating, we will obtain:

$$\ln p(t) = -\int_0^t \lambda(t)\,dt,$$

whence

$$p(t) = e^{-\int_0^t \lambda(t)\,dt}. \qquad (3.6)$$

Thus reliability is expressed as the failure rate.

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In the particular case when $\lambda(t) = \lambda$ = const, formula (3.6) gives:

$$p(t) = e^{-\lambda t}, \qquad (3.7)$$

i.e. the already known to us exponential law of reliability.

Using the form of the "flow of failures", it is possible to interpret not only formula (3.7), but also more common/general/total formula (3.6). Let us visualize (it is completely conditional!) that to cell/element with the arbitrary law of reliability p(t) functions the flow of failures with alternating/variable intensity $\lambda(t)$. Then

formula (3.6) for p(t) expresses probability that on the section of time (0, t) will appear not one failure.

in such a manner, both under exponential and under any other law of reliability the work of cell/element, beginning with the torque/moment of connection/inclusion t = 0, it is possible to visualize so that on cell/element functions the Poisson flow of failures; for the exponential law of reliability, this there will be flow with constant intensity λ, and for non-exponential - with alternating/variable intensity $\lambda(t)$.

Let us note that this form is suited only when the refused cell/element is not substituted new. If, as we this they made more earlily, to immediately substitute the refused cell/element new, the flow of failures at will not be Poisson. It is real/actual, its intensity will depend not simply on time t, past from the beginning of entire process, but also on time τ, the past from random torque/moment connection/inclusion of precisely this cell/element; that means the flow of events it has an aftereffect and Poisson it is not.

But if for the extent/elongation of entire process of the given element being investigated is not substituted and can refuse not more than one times, then during the description of the process, which

depends on its functioning, it is possible to use the pattern of the Markovian process, but at the alternating/variable, and constant intensity of flow of failures.

If the non-exponential law of reliability comparatively differs little from the exponential, then it is possible, for the purpose of simplification, to approximately replace it exponential (Fig. 7.12). The parameter λ of this law is chosen so as to preserve constant/invariable the mathematical expectation of the time of failure-free operation, equal as we we know that the area, limited curved p(t) and by the coordinate axes. For this, it is necessary to place the parameter λ of exponential law equal to

$$\lambda^* = \frac{1}{\bar{t}},$$

where to $\bar{t}$ - area, limited of the curve of reliability p(t).

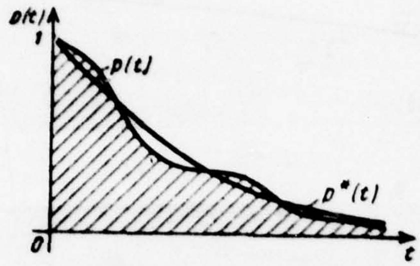

Fig. 7.12.

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Thus, if we wish to characterize the reliability of cell/element by certain average failure rate, it is necessary as this intensity to take the value, reciprocal to the mean time of the failure-free operation of cell/element.

Above we defined value $\bar{t}$ as area, limited curved p(t). However, if is required to know only mean time of the failure-free operation of cell/element, to simply find it directly from statistical material as arithmetic mean of all observed values of random variable T - operating time of cell/element to its failure. This method can be used also in the case when the number of experiments small and does not make it possible sufficiently accurately to construct the curve p(t).

Example 1. The reliability of cell/element p(t) decreases in the course of time according to linear law (Fig. 7.13). To find rate of failures $\lambda(t)$ and the mean time of the failure-free operation of cell/element $\bar{t}$.

Solution. On formula (3.4) on section $(0,\ t_0)$ we have:

$$\lambda(t)=\frac{f(t)}{p(t)}=-\frac{p'(t)}{p(t)}.$$

According to the assigned law of the reliability

$$p(t)=1-\frac{t}{t_0}\quad(0<t<t_0),$$

$$p'(t)=-\frac{1}{t_0},$$

$$\lambda(t)=\frac{1}{t_0\left(1-\frac{t}{t_0}\right)}=\frac{1}{t_0-t}.$$

Plotted function $\lambda(t)$ is shown on Fig. 7.14. When $t\to t_0$ $\lambda(t)\to\infty$. The mean time of failure-free operation is equal to the area, limited curved p(t) and by the axes of the coordinates (see Fig. 7.13): $\bar{t} = t_0/2$.

Example 2. The rate of failures of cell/element $\lambda(t)$ varies according to the law, presented in Fig. 7.15. To find the law of reliability p(t).

Solution. On section (0. 1)

$$\lambda(t)=3-2t.$$

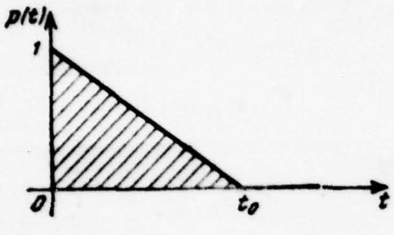

Fig. 7.13.

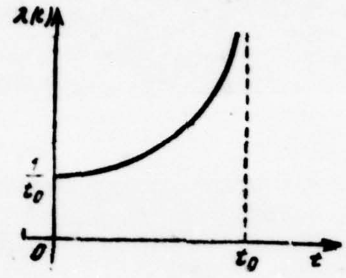

Fig. 7.14.

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On formula (3.6)

$$p(t)=e^{-\int_0^t \lambda(t)\,dt}=e^{-(3t-t^2)}.$$

Let us compute p(t) on section t > 1. In common/general/total formula (3.6) let us decompose the interval/gap of integration into two: from 0 to 1 and from 1 to t:

$$\int_0^t \lambda(t)\,dt=\int_0^1 \lambda(t)\,dt+\int_1^t \lambda(t)\,dt=\int_0^1 (3-2t)\,dt+\int_1^t dt=2+t-1=1+t;$$

$$p(t)=e^{-(1+t)}.$$

The graph of the law of reliability is shown on Fig. to 7.16. The shaded area represents the mean time of the failure-free operation:

$$\bar{t}=\int_0^1 e^{-(3t-t^2)}\,dt+\int_1^\infty e^{-(1+t)}\,dt.$$

The second integral is here equal to

$$-e^{-(1+t)}\Big|_1^\infty=e^{-2}=0{,}135.$$

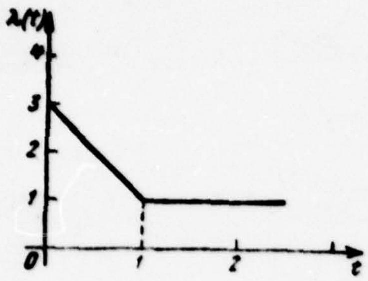

Fig. 7.15.

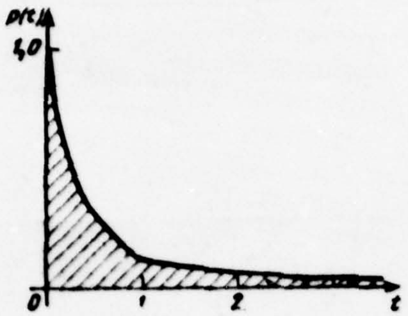

Fig. 7.16.

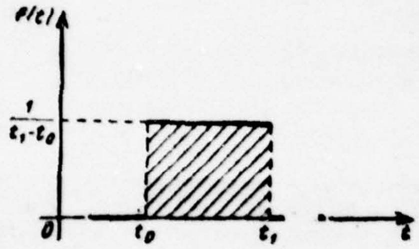

Fig. 7.17.

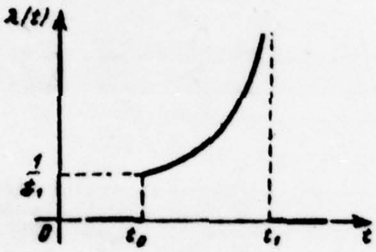

Fig. 7.18.

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As concerns the first, then it is calculated approximately (it is numerical):

$$\int_0^1 e^{-(3t-t^2)}\,dt \approx 0,370,$$

whence

$$\bar{T} \approx 0,370+0,135=0,505.$$

Example 3. The density of distribution of the time of the failure-free operation of cell/element is constant on section (t_1, t_2) and it is equal to zero out of this section (Fig. 7.17). To find rate of failures $\lambda(t)$.

Solution. We have:

$$\lambda(t)=\frac{f(t)}{p(t)}=\frac{f(t)}{1-q(t)}\quad (t_0<t<t_1),$$

where

$$q(t)=\int_{t_0}^{t}\frac{1}{t_1-t_0}\,dt=\frac{t-t_0}{t_1-t_0},$$

whence

$$\lambda(t)=\frac{1}{t_1-t}.$$

the graph of failure rate is shown on Fig. to 7.18; when $t\to t_1$ $\lambda(t)\to\infty$.

4. Determination of reliability of system from reliability of its cell/elements. Reliability of the nonredundant system.

Let certain technical system S be comprised from n of the cell/elements of units): $Э_1, Э_2, \ldots, Э_n$.

Let us assume that the reliability of cell/elements to us are known. Arises the question concerning the determination of the reliability of system. It depends on how cell/elements are united into system, which function of each of them and in which measure the exact work of each cell/element is necessary for the operation of system as a whole.

In a series of systems, the insufficient reliability of cell/elements rises because of their redundancy (redundancy). Redundancy lies in the fact that together with cell/element $Э_1$ into system is introduced spare (spare) the cell/element $Э_1'$, to which the system is changed over in the case of failure of basic cell/element. The number of spare cell/elements can be and more than one.

The simplest case in calculated sense is simple system (or system without redundancy). In this system the failure of any cell/element is equivalent to the failure of system as a whole by analogy with the chain/network of the series-connected conductors, the break of each of which is equivalent to breaking an entire circuit, we will call this connection/compound of cell/elements "consecutive" (Fig. 7.19). One should be specified that "consecutive" this connection/compound of cell/elements is to only in sense of

reliability, physically they can gear as conveniently.

It is expressed the reliability of the simple system through the reliability of its cell/elements. Let there be certain time interval (0, τ), during which it is required to ensure the failure-free operation of system.

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Then, if the reliability of system is characterized by the law of reliability P(t), to us it is important to know the value of this reliability with t = τ, i.e., P(τ). These are not function, but the specific number; let us reject/throw argument τ and will designate the reliability of system simply P. Let us analogously designate the reliability of separate cell/elements $p_1, p_2, \ldots, p_n$.

For the failure-free operation of simple system for a period of time τ, it is necessary that would work smoothly each of its cell/elements. Let us designate: S - event, which consists of the failure-free operation of system for time τ; $Э_1, Э_2, \ldots, Э_n$ - events, which consist of the failure-free operation of equivalent components. Event S is a product (coincidence) of the events $Э_1, Э_2, \ldots, Э_n$:

$$S = Э_1 \cdot Э_2 \cdot \ldots \cdot Э_n.$$

Let us assume that cell/elements $Э_1, Э_2, \ldots, Э_n$ they reject independently of each other (or as we will speak for brevity, "are independent on failures", and entirely briefly "are independent"). Then according to product rule of probabilities for the independent events

$$P(S) = P(Э_1)P(Э_2) \ldots P(Э_n),$$

or in other designations,

$$P = p_1 \cdot p_2 \cdot \ldots \cdot p_n, \qquad (4.1)$$

and it is shorter

$$P = \prod_{i=1}^{n} p_i, \qquad (4.2)$$

i.e. the reliability of the simple system, comprised of independent cell/elements, is equal to the product of the reliability of its cell/elements.

In the particular case when all cell/elements possess the identical reliability

$$p_1 = p_2 = \ldots = p_n = p,$$

formula (4.2) takes the form:

$$P = p^n. \quad (4.3)$$

Example 1. Simple system consists of 10 independent cell/elements, the reliability of each of which is equal p = 0.95. To determine the reliability of system.

Solution. On formula (4.3)

$$P = 0{,}95^{10} \approx 0{,}6.$$

From an example it is evident, as sharply falls the reliability of simple system with an increase in the number of cell/elements. if the number of cell/elements n is great, then for providing at least the acceptable reliability P of system each cell/element must possess very high reliability.

Let us raise the question: which reliability p it must possess separate cell/element, so that the system, comprised of n of such cell/elements, would possess the assigned reliability $\mathcal{P}$?

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Set/assuming in formula (4.3) $P = \mathcal{P}$, we will obtain:

$$p = \sqrt[n]{\mathcal{P}}. \qquad (4.4)$$

Example 2. Simple system consists of 1000 equally reliable, independent cell/elements. Such reliability must possess each of them, so that the reliability of system would be not less than 0.9?

Solution. On formula (4.4):

$$p = \sqrt[1000]{\mathcal{P}} = \sqrt[1000]{0{,}9}; \quad \lg p = \frac{1}{1000} \lg 0{,}9, \quad p \approx 0{,}9999.$$

It is expressed the rate of failures of simple system $\Lambda(t)$ through the rates of failures $\lambda_i(t)$ of its separate cell/elements. We have:

$$P(t) = \exp\left\{-\int_0^t \Lambda(t)\,dt\right\}^{*)},$$

$$p_i(t) = \exp\left\{-\int_0^t \lambda_i(t)\,dt\right\} \quad (i = 1, \ldots, n).$$

FOOTNOTE [1]. Here $\exp(x) = e^x$. ENDFOOTNOTE.

Let us substitute these expressions into formula (4.2); we will obtain:

$$\exp\left\{-\int_0^t \Lambda(t)\,dt\right\} =$$
$$\exp\left\{-\left[\int_0^t \lambda_1(t)\,dt + \int_0^t \lambda_2(t)\,dt + \ldots + \int_0^t \lambda_n(t)\,dt\right]\right\} =$$
$$= \exp\left\{-\int_0^t [\lambda_1(t) + \lambda_2(t) + \ldots + \lambda_n(t)]\,dt\right\}$$

or, it is shorter,

$$\exp\left\{-\int_0^t \Lambda(t)\,dt\right\} = \exp\left\{-\int_0^t \sum_{i=1}^n \lambda_i(t)\,dt\right\},$$

whence

$$\int_0^t \Lambda(t)\,dt = \int_0^t \sum_{i=1}^n \lambda_i(t). \tag{4.5}$$

Differentiating (4.5) on t, we will obtain:

$$\Lambda(t) = \sum_{i=1}^n \lambda_i(t), \tag{4.6}$$

i.e. with "consecutive" sections of the independent cell/elements with intensity of failures, they store/add up.

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This and it is logical, since for a simple system the failure of cell/element is equivalent to failure system, which means, that all flows of the failures of separate cell/elements combine into one flow of the failures of system with the intensity, equal to the sum of the intensities of separate flows.

Example 3. The simple system S consists of three independent cell/elements $Э_1$, $Э_2$, $Э_3$ (Fig. 7.20), the densities of distribution of the time of failure-free operation of which are assigned by the formulas:

$$\left.\begin{array}{l} f_1(t)=1, \\ f_2(t)=2t, \\ f_3(t)=2(1-t) \end{array}\right\} \text{with} \quad 0<t<1$$

(Fig. 7.21-7.23). To find the rate of failures of system.

Solution. We determine the unreliability of each cell/element:

$$\left.\begin{array}{l} q_1(t)=t, \\ q_2(t)=t^2, \\ q_3(t)=2t-t^2 \end{array}\right\} \text{with} \quad 0<t<1.$$

Hence the reliability of the cell/elements:

$$\left.\begin{array}{l} p_1(t)=1-t, \\ p_2(t)=1-t^2, \\ p_3(t)=1-2t+t^2 \end{array}\right\} \text{with} \quad 0<t<1.$$

Fig. 7.20.

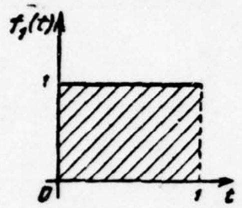

Fig. 7.21.

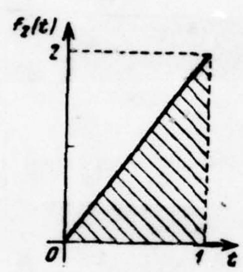

Fig. 7.22.

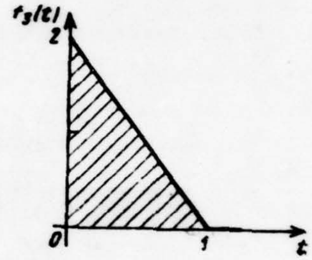

Fig. 7.23.

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Rates of failures of the cell/elements:

$$\left.\begin{aligned} \lambda_1(t) &= \frac{1}{1-t}, \\ \lambda_2(t) &= \frac{2t}{1-t^2}, \\ \lambda_3(t) &= \frac{2(1-t)}{1-2t+t^2} = \frac{2}{1-t}. \end{aligned}\right| \text{ with } \quad 0<t<1.$$

Store/adding up, we have:

$$\Lambda(t) = \lambda_1(t) + \lambda_2(t) + \lambda_3(t) =$$
$$= \frac{1}{1-t} + \frac{2t}{1-t^2} + \frac{2}{1-t} = \frac{3+5t}{1-t^2}.$$

5. Reliability of the redundant system (" hot reserve").

One of the ways of the increase of the reliability of system is the introduction into it of the duplicating (spare) cell/elements. Spare cell/elements are included in system as "in parallel" by the fact whose reliability is insufficient.

Let us consider a simplest example of the redundant system: two "in parallel" connected of cell/element $Э_1$ and $Э_2$ (Fig. 7.24). Works at first "basic" cell/element $Э_1$; if it refused, system automatically is changed over to "reserve" cell/element $Э_2$. Let us assume that the cell/elements $Э_1$ and $Э_2$ are independent on failures and that their reliability (probability of failure-free operation) for the which interests us time $t = \tau$ are equal to respectively p_1 and p_2. Let us assume also that the reliability of the second cell/element does not depend on that, was included this cell/element in work for time τ and when it was included. This picture is observed, for example, if cell/element $Э_2$ regardless of the fact, works it or not, it is held under operating voltage/stress (so called ("hot reserve").

Let us determine under these conditions the reliability of the redundant system S. Let us pass to the probability of opposite event - failure of system S. Let us designate the failure of system $\bar{S}$. So that the event $\bar{S}$ would occur, is necessary, that they would refuse both the cell/element: and the first and the second:

$$\bar{S} = \bar{S}_1 \bar{S}_2.$$

Hence according to product rule of the probabilities of the independent events:

$$P(\bar{S}) = P(\bar{S}_1)\,P(\bar{S}_2).$$

Designating the unreliability of system Q, and the unreliability of cell/elements q_1, q_2, we will obtain:

$$Q = q_1 q_2, \qquad (5.1)$$

i.e. during the "parallel" connection/compound of the independent cell/elements of their unreliability, they are multiplied.

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Passing in formula (5.1) from unreliability to reliability, we have

$$1 - P = (1 - p_1)(1 - p_2),$$

whence

$$P = 1 - (1 - p_1)(1 - p_2). \qquad (5.2)$$

With the arbitrary number n of the which duplicate each other independent cell/elements, the reliability of block from such cell/elements (Fig. 7.25) is computed from the formula

$$P=1-(1-p_1)(1-p_2)\ \dots\ (1-p_n), \tag{5.3}$$

or, it is shorter,

$$P=1-\prod_{i=1}^{n}(1-p_i). \tag{5.4}$$

In the particular case when the reliability of all cell/elements are identical:

$$p_1=p_2=\dots=p,$$

formula (5.4) takes the form:

$$P=1-(1-p)^n. \tag{5.5}$$

Example 1. The protecting device, which ensures the safety of work with materiel, consists of three duplicating each other of safety device/fuses. Reliability of each of them p = 0.9. Safety device/fuses are independent in the sense of reliability. To find the reliability of entire device.

Solution. On formula (5.5)

$$P=1-(1-0{,}9)^3=0{,}999.$$

Until now, speaking about "changeover" to spare cell/element, we assumed that either for this is not required special switching system

(as in the case with safety device/fuses), or the reliability of switching system is equal to unit.

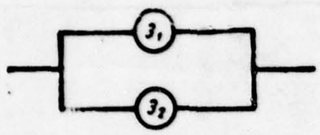

Fig. 7.24.

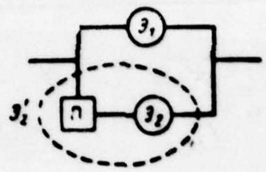

Fig. 7.25.

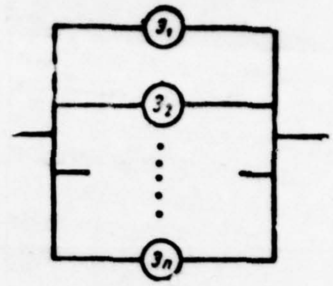

Fig. 7.26.

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If this not then, then it is easy to take into account its incomplete reliability.

Let us assume that the block consists of two "in parallel" connected cell/elements $Э_1$ and $Э_2$ (Fig. 7.26). In the case when cell/element $Э_1$ goes out of order, switching system $П$ system to another, spare cell/element $Э_2$. The reliability of cell/elements $Э_1$, $Э_2$ and of switch $П$ are equal to respectively p_1, p_2 and $p_П$. Let us determine the reliability of entire block. For this, is joined the switch $П$ and cell/element $Э_2$ into one "consecutive" circuit with the reliability

$$p_2' = p_П p_2.$$

Considering this chain/network as one in parallel connected conditional cell/element $Э'_2$, let us find from formula (5.2) the reliability of the block:

$$P=1-(1-p_1)(1-p_2')=1-(1-p_1)(1-p_n p_2). \quad (5.6)$$

Thus, the incomplete reliability cf switch can be taken into account by the simple multiplication of the reliability of spare cell/element by the reliability of switch.

If spare cell/elements not one, but it is more: $Э_2, \ldots, Э_n$ and each of them is furnished by their switch with reliability respectively $p_n^{(2)}, p_n^{(3)}, \ldots, p_n^{(n)}$, then in formula (5.3) it is necessary to multiply the reliability of each spare cell/element by the reliability of the switch:

$$P=1-(1-p_1)(1-p_n{}^{(2)}p_2)(1-p_n{}^{(3)}p_3)\ldots(1-p_n{}^{(n)}p_n). \quad (5.7)$$

It can seem that the changeover to any spare cell/element is realized one and the same switch П (Fig. 7.27). Then switch П together with entire block of spare cell/elements can be considered as one conditional cell/element $Э'_2$ with reliatility p'_2, equal to

$$p_2'=p_n[1-(1-p_2)(1-p_3)\ldots(1-p_n)], \quad (5.8)$$

a the reliability of entire blcck will be computed according to the formula

$$P=1-(1-p_1)(1-p_2'). \quad (5.9)$$

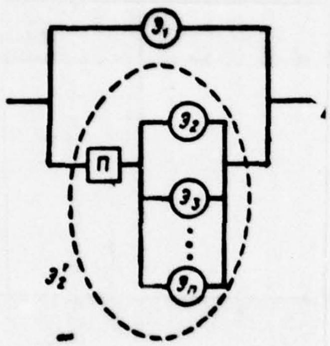

Fig. 7.27.

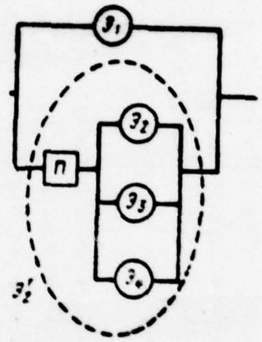

Fig. 7.28.

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Example 2. To determine the reliability of the block, which consists of basic cell/element $Э_1$ with reliability $p_1 = 0.9$ and three spare cell/elements: $Э_2$, $Э_3$, $Э_4$, having that one ~~all the~~ reliability:

$$p_2 = p_3 = p_4 = 0,9.$$

Switching to spare cell/elements in the case of failure of any of the cell/elements is realized with the help of one and the same switch, which has reliability $p_п = 0,95$ (Fig. 7.28). To find the reliability of block.

Solution. Is joined switch with spare cell/elements E_2, E_3, E_4

into conditional cell/element $Э'_2$ with the reliability

$$p_2' = p_п [1-(1-p_2)(1-p_3)(1-p_4)] = 0,95(1-0,1^3) \approx 0,949.$$

Reliability of entire block:

$$P = 1-(1-0,9)(1-0,949) \approx 0,995.$$

Let us note that in this example comparatively low reliability of switch virtually depreciates a large quantity (three!) of spare cell/elements. The considerably greater reliability of system we would obtain, if each cell/element was furnished by its switch:

$$P' = 1-(1-0,9)(1-0,95\cdot 0,9)^3 \approx 0,9997.$$

Until now, we examined the systems, duplicating one basic cell/element. In the general case in the redundant systems, can be applied both "the consecutive" and "parallel" connection/compounds of cell/elements, moreover as a rule, are doubled least reliable cell/elements. During the estimation of the reliability of this system, it is necessary to dismember it to a series of the "subsystems", which do not have common cell/elements, to find the reliability of each of them and, considering subsystems as conditional cell/elements, to consider the reliability of system as a whole.

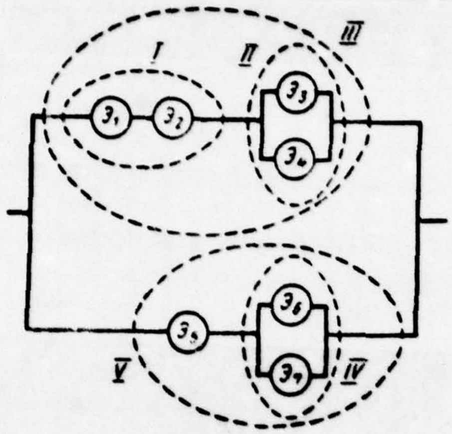

Fig. 7.29.

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Example 3. To determine the reliability of the system, which consists of cell/elements Э_1, Э_2, ..., Э_7 with reliability p_1, p_2, ..., p_7 (Fig. 7.29).

Solution. Subsystem I - "consecutively" connected cell/elements Э_1 and Э_2; the reliability:

$$P_I = p_1 p_2$$

Subsystem II - "in parallel" connected cell/elements Э_3 and Э_4: the reliability:

$$P_{II} = 1-(1-p_3)(1-p_4).$$

Subsystem III - "consecutively" connected I and II; the reliability:

$$P_{III}=p_I P_{II}.$$

Subsystem IV - "in parallel" connected $Э_6$ and $Э_7$; the reliability:

$$P_{IV}=1-(1-p_6)(1-p_7).$$

Subsystem V - "consecutively" connected $Э_5$ and IV; the reliability:

$$P_V=p_5 P_{IV}.$$

Entire system - "in parallel" connected III and V; the reliability:

$$P=1-(1-P_{III})(1-P_V).$$

6. Reliability of the redundant system (" cold" and the "lightened" reserve).

To the these on we examined only the case when the reliability

cf each duplicating cell/element on depends on that, when was include/connected in work this cell/element. This case which we conditionally named "hot redundancy", idle time itself of all possible. Is much more complex the case, when spare cell/element before its starting not at all can reject ("cold" redundancy) or can reject, but with other, by less probability density, than after connection/inclusion (the "lightened" redundancy).

In the examination of the tasks, connected with the cold or lightened redundancy, to us insufficient will introduce the reliability of system and cell/elements for one, previously fix/recorded, the values of time τ; will have to analyze entire random process of the functioning of system.

Let us consider several tasks, which relate to the cold and lightened redundancy.

Task 1. General case of the calculation of the reliability of the redundant system (the "lightened" or "cold" reserve). System (block) consists of "in parallel" connected cell/elements $Э_1$ and $Э_2$ (basic and spare). The intensity of flow of the failures of the first cell/element $\lambda_1(t)$; with the failure of the first cell/element occurs automatic and reliable changeover to spare $(p_n = 1)$. The intensity of flow of the failures of spare cell/element before its inclusion into

work $\lambda_2(t)$ (cell/element works in the "lightened" conditions/mode).

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After its starting, at the moment of the failure of the first cell/element, the intensity instantly runs up (Fig. 7.30) and becomes equal to the intensity $\tilde{\lambda}_2$, which it is logical to assume not only on the current time t depending, but also from that period t_1, during which the cell/element operated in the lightened conditions/mode:

$$\tilde{\lambda}_2 = \lambda_2(t/t_1).$$

It is required to find the reliability of system P(t).

Let us consider the set of two random variables:

T_1 - torque/moment of the failure of basic cell/element,

T_2 - torque/moment of the failure of spare cell/element.

Event A (failure-free operation of system to torque/moment t - lies in the fact that at least one of values T_1, T_2 will take value, is larger than t (at least one cell/element will work up to torque/moment t). The probability of opposite event - the failure of system to torque/moment t - will be

$$P(\overline{A}) = 1 - P(A) = P(T_1 < t,\ T_2 < t).$$

Let us find the combined density of distribution of random variables T_1 and T_2, designating its $f(t_1, t_2)$. Random variables T_1, T_2 are dependent, and

$$f(t_1, t_2) = f_1(t_1) f(t_2 | t_1), \tag{6.1}$$

where $f_1(t_1)$ - unconditional density of distribution of value T_1, $f(t_2/t_1)$ - conditional density of distribution of value T_2 (when value T_1 it took value t_1).

Let us find both densities. On formula (3.4) §3

$$f_1(t_1) = \lambda_1(t_1) p_1(t_1),$$

where $p_1(t_1)$ - the reliability of cell/element Э$_1$, by the force of formula (3.6) it is equal to

$$p_1(t_1) = \exp\left\{ -\int_0^{t_1} \lambda_1(t)\, dt \right\}.$$

Hence

$$f_1(t_1) = \lambda_1(t_1) \exp\left\{ -\int_0^{t_1} \lambda_1(t)\, dt \right\}. \tag{6.2}$$

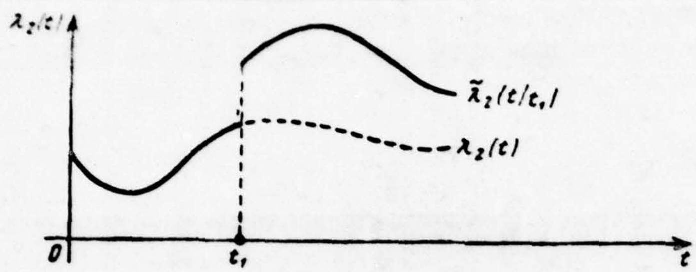

Fig. 7.30.

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Let us find the conditional density $f(t_2/t_1)$. The conditional rate of failures of spare cell/element when $T_1 = t_1$, will be:

$$\lambda_2(t_2/t_1) = \begin{cases} \lambda_2(t_2) & \text{with } t_2 < t_1, \\ \tilde{\lambda}_2(t_2/t_1) & \text{with } t_2 > t_1. \end{cases} \quad (6.3)$$

At this intensity let us find the conditional density of distribution of the time of the failure-free operation of the spare cell/element:

$$f(t_2/t_1)=\begin{cases}\lambda_2(t_2)\exp\left\{-\int_0^{t_2}\lambda_2(t)\,dt\right\} & \text{при} \quad t_2<t_1,\\ \tilde{\lambda}_2(t_2/t_1)\exp\left\{-\int_0^{t_1}\lambda_2(t)\,dt-\int_{t_1}^{t_2}\tilde{\lambda}_2(t/t_1)\,dt\right\} & \text{при} \quad t_2>t_1.\end{cases} \quad (6.4)$$

Key: (1). with.

Thus, the combined density of distribution of the system of random variables T_1, T_2 is found:

$$f(t_1,t_2)=f_1(t_1)f(t_2/t_1)=$$

$$\begin{cases}\lambda_1(t_1)\lambda_2(t_2)\exp\left\{-\int_0^{t_1}\lambda_1(t)\,dt-\int_0^{t_2}\lambda_2(t)\,dt\right\} & \text{при} \quad t_2<t_1,\\ \lambda_1(t_1)\tilde{\lambda}_2(t_2/t_1)\exp\left\{-\int_0^{t_1}\lambda_1(t)\,dt-\int_0^{t_1}\lambda_2(t)\,dt-\int_{t_1}^{t_2}\tilde{\lambda}_2(t/t_1)\,dt\right\} & \text{при} \quad t_2>t_1.\end{cases} \quad (6.5)$$

Key: (1). with.

Knowing this combined density, it is possible to find the failure probability of system to torque/moment t:

$$P(\bar{A})=P(T_1<t,\ T_2<t)=$$

$$=\int_0^t\int_0^t f(t_1,t_2)\,dt_1\,dt_2,$$

whence the unknown reliability of system:

$$P(t)=1-\int_0^t\int_0^t f(t_1,t_2)\,dt_1\,dt_2. \quad (6.6)$$

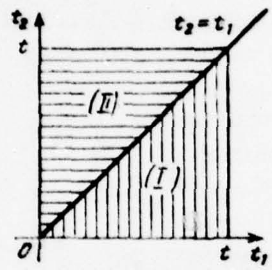

Fig. 7.31.

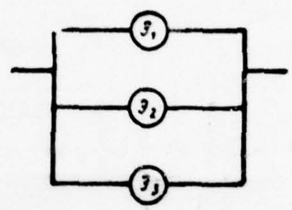

Fig. 7.32.

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During calculation on formulas (6.5)-(6.6) it is necessary to keep in mind, what expression of function $f(t_1/t_2)$ is dissimilar by one and another side from the straight line $t_2 = t_1$ - bisectrix of the first quardant (Fig. 7.31). Ranges of integration in Fig. 7.31 are noted by different shading. In range I function $f(t_1/t_2)$ is expressed by the first of formulas (6.5), in range II, - the second; consequently,

$$P(t)=1-\left\{\iint\limits_{(1)}\lambda_1(t_1)\lambda_2(t_2)\times\right.$$

$$\times\exp\left\{-\int\limits_0^{t_1}\lambda_1(t)\,dt-\int\limits_0^{t_2}\lambda_2(t)\,dt\right\}dt_1\,dt_2+$$

$$+\iint\limits_{(1')}\lambda_1(t_1)\tilde{\lambda}_2(t_2/t_1)\times$$

$$\times\exp\left\{-\int\limits_0^{t_2}\lambda_1(t)\,dt-\int\limits_0^{t_1}\lambda_2(t)\,dt-\int\limits_{t_1}^{t_2}\tilde{\lambda}_2(t/t_1)\,dt\right\}dt_1\,dt_2. \qquad (6.7)$$

With the assigned concrete/specific/actual form of the function $\lambda_1(t)$, $\lambda_2(t)$, $\tilde{\lambda}_2(t/t_1)$ integral (6.7) can be calculated, in the simplest cases analytically, more frequent - numerically.

Let us note that the found by us solution of the problem of estimating the reliability for the case of the "lightened" reserve is related also to the case of "cold" reserve - in this case $\lambda_2(t) = 0$, so that in formula (6.7) there remains only one integral - the second, yes even that also will be simplified.

We see that in the case even of the one spare cell/element, working in the lightened (or cold) reserve the task of estimating the reliability of system is sufficiently complex. But if the number of spare cell/elements is more than one, task even more is complicated.

However, task can be highly simplified, if one assumes that the flows of malfunctions, which function on all cell/elements (basic and spare), represent by themselves simplest flows, the intensity of each of which is constant (this assumption is equivalent to the fact that the law of the reliability of each cell/element - exponential, and starting cell/element varies only the parameter of this law). With this assumption the reliability of system **S** can be found by the method of solution of differential equations for the probabilities of its states.

Task 2. System with cold reserve and the simplest flows of failures. The redundant system (block) **S** consists of basic cell/element $\mathbf{Э}_1$ and two spare: $\mathbf{Э}_2$, $\mathbf{Э}_3$. With the failure of cell/element, $\mathbf{Э}_1$ in work is included $\mathbf{Э}_2$, with the failure of $\mathbf{Э}_2$-$\mathbf{Э}_3$ (Fig. 7.32).

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Before connection/inclusion each of the spare cell/elements is located and "cold" reserve and refuse it cannot. The intensity of flow of the failures of basic cell/element λ_1; the intensity of flow of the failures of each of the spare cell/elements when they work, is identical and equal to λ_2. All flows of failures are simplest. It is required to determine the reliability of system **S**.

Let us present the process, which takes place in system S as Markovian process (see Chapter 4) with continuous time and with discrete states:

S_1 - works basic cell/element $Э_1$,

S_2 - works spare cell/element $Э_2$,

S_3 - works spare cell/element $Э_3$,

S_4 - works not one cell/element.

The graph/count of the states of system is shown on Fig. 7.33. Since the restoration/reductions of cell/elements does not occur, all arrow/pointers on graph/count conduct to one side.

The system of equations of Kolmogorov for the probabilities of states will be:

$$\left.\begin{aligned}
\frac{dp_1}{dt} &= -\lambda_1 p_1,\\
\frac{dp_2}{dt} &= -\lambda_2 p_2 + \lambda_1 p_1,\\
\frac{dp_3}{dt} &= -\lambda_3 p_3 + \lambda_2 p_2,\\
\frac{dp_4}{dt} &= \lambda_3 p_3.
\end{aligned}\right\} \qquad (6.8)$$

To them it is necessary to adjoin the normalizing condition:

$$p_1 + p_2 + p_3 + p_4 = 1. \qquad (6.9)$$

From the first equation we express p_1 as function t:

$$p_1(t) = e^{-\lambda_1 t} \qquad (6.10)$$

(initial condition by which we integrated this equation, $p_1(0) = 1$). Substituting (6.10) in the second equation, we will obtain:

$$\frac{dp_2}{dt} = -\lambda_2 p_2 + \lambda_1 e^{-\lambda_1 t}. \qquad (6.11)$$

Let us integrate this equation with the initial condition $p_2(0) = 0$; we will obtain:

$$p_2(t) = \frac{\lambda_1}{\lambda_2 - \lambda_1} e^{-\lambda_1 t} - \frac{\lambda_1}{\lambda_2 - \lambda_1} e^{-\lambda_2 t}. \qquad (6.12)$$

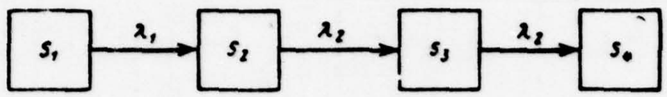

Fig. 7.33.

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This function let us substitute into third equation (6.8) ; we will obtain:

$$\frac{dp_3}{dt} = -\lambda_2 p_3 + \frac{\lambda_1 \lambda_2}{\lambda_2-\lambda_1} e^{-\lambda_1 t} - \frac{\lambda_1 \lambda_2}{\lambda_2-\lambda_1} e^{-\lambda_2 t}. \qquad (6.13)$$

Equation (6.13) it is necessary to integrate also under the initial condition $p_3(0) = 0$; we will obtain:

$$p_3(t) = \frac{\lambda_1 \lambda_2}{(\lambda_2-\lambda_1)^2} e^{-\lambda_1 t} - \frac{\lambda_1 \lambda_2}{(\lambda_2-\lambda_1)^2} e^{-\lambda_2 t} - \frac{\lambda_1 \lambda_2 t}{\lambda_2-\lambda_1} e^{-\lambda_2 t}. \qquad (6.14)$$

For the determination of function $p_4(t)$ it is not necessary to integrate last/latter equation (6.8) - it it is possible to find from condition (6.9):

$$p_4(t) = 1 - P(t) = 1 - (p_1(t) + p_2(t) + p_3(t)) =$$
$$= 1 - \frac{\lambda_2^2}{(\lambda_2-\lambda_1)^2} e^{-\lambda_1 t} - \left[\frac{\lambda_1^2 - 2\lambda_1 \lambda_2}{(\lambda_2-\lambda_1)^2} - \frac{\lambda_1 \lambda_2 t}{\lambda_2-\lambda_1}\right] e^{-\lambda_2 t}.$$

Task 3. System with the lightened reserve and the simplest flows cf failures. The redundant system (block) S consists of basic cell/element $Э_1$ and three spare: $Э_2$, $Э_3$, $Э_4$ (Fig. 7.34). Basic cell/element undergoes the simplest flow of failures with intensity λ_1; each of the spare before their connection/inclusion undergoes the flow of failures with intensity λ_2; after the connection/inclusion of spare cell/element, this intensity instantaneously runs up to value $\tilde{\lambda}_2$. With the failure of basic cell/element, $Э_1$ is included in work spare $Э_2$, with the failure of $Э_2$-$Э_3$ and i.e.

It is required to determine the reliability of system.

Let us label the states of system by two indices: the first is equal to unit, if basic cell/element works, and zero - if on works: the second is equal to the number of exact spare cell/elements:

S_{13} - basic cell/element is exact (it works), everything three spare exact;

S_{12} - basic cell/element it is exact (it works), of three spare one it refused, two were exact;

S_{11}, basic cell/element is exact (it works), of three spare two they refused, one was exact;

S_{10} - basic cell/element it is exact (it works), everything three spare refused;

S_{03} - basic cell/element it refused, works one of the spare, remaining two are exact;

S_{02} - basic cell/element it refused, works one of spare, of the others spare one is exact, another refused;

S_{01} - basic cell/element it refused, operates one of the spare, the others two spare refused;

S_{00} - all cell/elements they refused.

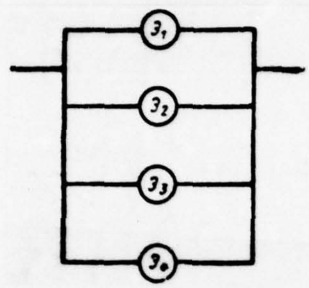

Fig. 7.34.

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The graph/count of the states of system is shown on Fig. to 7.35.

System of equations Kolmogorov a for the probabilities of states takes the form:

$$\left.\begin{aligned}
\frac{dp_{13}}{dt} &= -(3\lambda_2 + \lambda_1)\, p_{13}, \\
\frac{dp_{12}}{dt} &= -(2\lambda_2 + \lambda_1)\, p_{12} + 3\lambda_2\, p_{13}, \\
\frac{dp_{11}}{dt} &= -(\lambda_2 + \lambda_1)\, p_{11} + 2\lambda_2\, p_{12}, \\
\frac{dp_{10}}{dt} &= -\lambda_1\, p_{10} + \lambda_2\, p_{11}, \\
\frac{dp_{03}}{dt} &= -(\lambda_2 + 2\lambda_2)\, p_{03} + \lambda_1\, p_{13}, \\
\frac{dp_{02}}{dt} &= -(\tilde{\lambda}_2 + \lambda_2)\, p_{02} + \lambda_1\, p_{12} + (\tilde{\lambda}_2 + 2\lambda_2)\, p_{03}, \\
\frac{dp_{01}}{dt} &= -\tilde{\lambda}_2\, p_{01} + \lambda_1\, p_{11} + (\tilde{\lambda}_2 + \lambda_2)\, p_{02}, \\
\frac{dp_{00}}{dt} &= \lambda_1\, p_{10} + \tilde{\lambda}_2\, p_{01}.
\end{aligned}\right| \qquad (6.15)$$

To these equations it is necessary to supplement condition:

$$p_{13} + p_{12} + p_{11} + p_{10} + p_{03} + p_{02} + p_{01} + p_{00} = 1,$$

making it possible to reject/throw any of equations (6.15).

The integration of system (6.15) can be realized in the following order: from the first equation we find $p_{13}(t)$:

$$p_{13}(t) = e^{-(3\lambda_2 + \lambda_1)t}. \qquad (6.16)$$

This expression is substituted in the second equation which now contains only one unknown function $p_{12}(t)$; we find it, we substitute in the third equation, and so on.

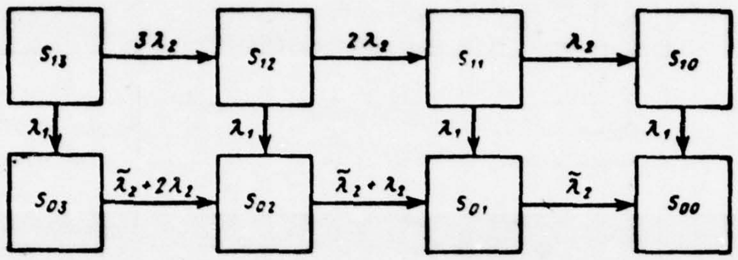

Fig. 7.35.

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At each step/pitch of this process the new functions we express as already known, thus far finally it is not reached p_{00}, which we express as all others:

$$p_{00}(t)=1-(p_{13}(t)+p_{12}(t)+p_{11}(t)+p_{10}(t)+ \\ +p_{03}(t)+p_{02}(t)+p_{01}(t)).$$

After calculations are produced and functions $p_{13}(t)$, ..., $p_{00}(t)$ found, it is possible to find the reliability of system P(t). It is obvious, it is equal to the sum of the probabilities of all states, in which the system works:

$$P(t)=p_{13}(t)+p_{12}(t)+p_{11}(t)+p_{10}(t)+ \\ +p_{03}(t)+p_{02}(t)+p_{01}(t).$$

cr, which is the same thing,

$$P(t)=1-p_{00}(t). \quad (6.17)$$

7. Reliability of system with restoration/reduction.

Until now, examining the tasks of reliability, we proceeded from the fact that the refused cell/element goes out of order finally and no restoration/reduction of its functicns it is produced. Is of interest research of the tasks of reliatility on the assumption that the refused cell/elements are restored - they are instantly substituted new or they are overhauled.

During the solution of this type of tasks, we will assume that all flows of events, which translate system from state into state, simplest (otherwise we such tasks will not manage).

Let us preliminarily do the following observation: all processes, connected with the reliability of the systems which we examined, until now, it was substantially unsteady; since the restoration/reductions of cell/elements was not, it is logical that with t $\rightarrow\infty$ the reliability of system vanished, and the "maximum conditions/mode" of system it was simply "it dces not work".

In tasks with restoration/reduction us they will interest not only transient processes in system, but also the steady-state conditions/modes, attained with $t \rightarrow \infty$. In this paragraph we will consider several tasks of the range of the reliability of systems with restoration/reduction.

Task 1 (task of spare cell/elements).

Works the simple system, which consists of one cell/element Э, which undergoes the simplest flow of failures with intensity λ. With failure the cell/element is instantly substituted new with the same characteristics. Available is N of spare cell/elements, which are located in "cold" reserve. To determine probability that this number of spare cell/elements to us will suffice for operational provisions of system for a period of time t (in other words, to find reliability P(t) of system with restoration/reduction).

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Solution. It is not difficult to note that stated problem is equivalent to the task of estimating the reliability of the redundant system with N by the spare cell/elements, working in cold reserve and, as such, it can be solved by the methods, proposed above. But we is solved by its somewhat different, simpler method.

Let us consider on axis 0t the "flow of restoration/reductions", i.e., the sequence of the moments of time at which they go out of order and instantly are restored cell/elements (Fig. 7.36). It is obvious, this - the simplest flow with intensity λ. The reliability of system P(t) is probability that up to torque/moment t the system will work. For this, it is necessary that on the section (0, t) it would refuse not more than N cell/elements (one basic and N-1 spare).

We know (see §4 chapters 4) that the number of events of the simplest flow, which fall to section with a length of t, is distributed according to the law of Poisson:

$$P_m = \frac{a^m}{m!} e^{-a},$$

where a = λt, i.e.,

$$P_m = \frac{(\lambda t)^m}{m!} e^{-\lambda t} \quad (m = 0, 1, \ldots). \tag{7.1}$$

Let us find probability that the number of points (events), that fall to section (0, t), will be not more than N. This probability will be the reliability of the system

$$P(t) = P_0 + P_1 + \ldots + P_N,$$

or, it is shorter,

$$P(t) = \sum_{m=0}^{N} P_m. \tag{7.2}$$

Substituting (7.1) in (7.2), we will obtain:

$$P(t)=\sum_{m=0}^{N}\frac{(\lambda t)^m}{m!}e^{-\lambda t}, \qquad (7.3)$$

or, after taking out $e^{-\lambda t}$ as the sign of sum,

$$P(t)=e^{-\lambda t}\sum_{m=0}^{N}\frac{(\lambda t)^m}{m!}. \qquad (7.4)$$

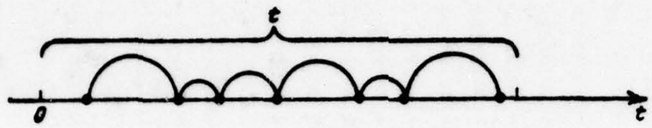

Fig. 7.36.

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Calculations on formulas (7.5) or (7.4) it is convenient to produce, using the tables of Poisson distribution P_m (or probabilities $R_m = 1 - \sum_{n=0}^{m-1} P_n$, which somewhat more conveniently are tabulated).

In Appendix (table 2) gives delays from the tables of Poisson distribution (probability P_m).

Example 1. Is examined the work of cell/element with restoration/reduction (task 1); the intensity of flow of failures λ = 2 (failure in hour). In our disposition N = 6 spare cell/elements. To determine the reliability of system P(t) in the function of time to τ = 5 hour (maximum operating time).

Solution. We will use Table 2 application/appendices. The first column of the table where P_7 is excellently from zero - these

are the column, which corresponds to a = 1, i.e., t = 0.5. Set/assuming t = 0.5 and store/adding up all probabilities for m > 6 (of them is different from zero only P_7), we obtain:

$$P(0,5) = 1 - 0,0001 = 0,9999.$$

For t = 1 (a = 2) we have:

$$P(1) = 1 - (0,0037 + 0,0009 + 0,0002) = 1 - 0,0048 = 0.9952 \approx 0,995.$$

For t = 2 (a = 4):

$$P(2) = 1 - (0,0595 + 0,0298 + 0,0132 + 0,0053 + 0.0019 + 0,0006 + 0,0002 + 0,0001) = 1 - 0,1106 \approx 0,889.$$

For t = 3 (a = 6) already more convenient to not pass to opposite event, but to compute probability that the number of failures will be lesser than seven:

$$P(3) = 0,0025 + 0,0149 + 0,0466 + 0,0892 + 0,1339 + 0,1606 + 0,1606 \approx 0,608$$

For t = 4 (a = 8):

$$P(4) = 0,0003 + 0,0027 + 0,0107 + 0,0286 + 0,0572 + 0,0916 + 0,1221 \approx 0,313.$$

For t = 5 (a = 10):

$$P(5) = 0,0000 + 0,0005 + 0,0023 + 0,0076 + 0,0189 + 0,0378 + 0.0631 \approx 0,130.$$

It is applied the obtained values for graph (Fig. 7.37).

Task 2. System consists not of one as of task 1, but of several cell/elements; among them

n_1 элементов группы 1,
n_2 элементов группы 2,
. . . .
n_k элементов группы k.

Key: (1). the cell/elements of group.

Each of the cell/elements of any group, independent of others, it can reject; the intensity of flow of failures for the cell/elements of different groups is equal respectively: $\lambda_1, \lambda_2, ..., \lambda_k$.

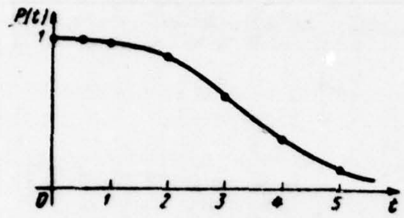

Fig. 7.37.

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All flows of failures - simplest. The refused cell/element is immediately substituted new. In supply there is N_1, N_2, ..., N_k the cell/elements of the corresponding groups. The absence of spare cell/element with next failure indicates the failure of device. It is required to determine the reliability of system P(t).

Solution. Since the absence of the spare cell/element of any group is equivalent to the failure of device, let us consider groups as "consecutively" connected cell/elements; then the reliability of system is equal to the product of the reliability of all groups. The reliability of the i group is defined as in task 1:

$$P^{(i)}(t) = e^{-\lambda_i t} \sum_{m=0}^{N_i} \frac{(\lambda_i t)^m}{m!}. \qquad (7.5)$$

Multiplying these reliability, we will obtain the reliability of the system:

$$P(t)=P^{(1)}(t)\cdot P^{(2)}(t)\cdot \ldots \cdot P^{(k)}(t),$$

or, it is shorter,

$$P(t)=\prod_{i=1}^{k} P^{(i)}(t). \tag{7.6}$$

Let us note that, using the brought out formulas, it is possible to not only consider the reliability of system with the assigned number of spare cell/elements, but also to determine that how many spare cell/elements it is necessary to have available so that the system with assigned t would have the specific reliability.

Example 2. To determine the number of spare cell/elements N which must be had available so that the system, which consists of one basic cell/element and N of spare with the intensity of flow of failures $\lambda = 0.5$, would have with $t = 8$ reliability not less than 0.95.

Solution. We have $a = \lambda t = 4$. In column Table 2 application/appendices, appropriate $a = 4$, we store/add up all probabilities, beginning with the latter, until sum reaches

$$1-0{,}95=0{,}05.$$

We obtain:

$$0,0001 + 0,0002 + 0,0006 + 0,0019 + 0,0053 + 0,0132 + 0,0298 = 0,0511.$$

Thus, probability that the number of refused cell/elements will be more than seven, is equal to 0.0511, i.e., N = 7 does not satisfy our requirement; but if we take N = 8, then the probability of the deficiency of cell/elements will be less than 0.05:

$$0,0001 + 0,0002 + 0,0006 + 0,0019 + 0,0053 + 0,0132 = 0,0213.$$

Hence, the number of spare cell/elements, which satisfies the condition of task, N = 8.

~~Vo~~ In all examined above tasks the restoration/reduction of cell/element occurred instantly; now we will consider the task where it is detained.

Task 3 (system of one cell/element with the delayed restoration/reduction).

System consists of one cell/element Э_1, the locating under action simplest flow of failures with intensity λ. The refused cell/element immediately begins to be restored (to be overhauled).

Flow of restoration/reductions - simplest, with intensity μ. The supply of resources for a repair is not limited.

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It is required to determine:

- generalized reliability of system $P(t)$ - probability that in torque/moment t the system will work;

- limiting value by generalized p - probability that in the arbitrary, sufficiently distant from beginning torque/moment the system will work;

- probability $\tilde{P}(t)$ of the fact that to the specific torque/moment the system will work generally smoothly (i.e. will be not one outage for restoration/reduction).

Solution. The states of system (in this case of cell/element) will be:

S_0 - works,

S_1 - is restored.

The graph/count of states is shown on Fig. 7.38. Equate/comparing the graph/count of states 7.38 with the graph/count of the states of the single-channel system of mass maintenance with the failures (see §3 chapters 5, Fig. 5.1), we see that they coincide; that means they coincide and the probability of states, i.e.,

$$\left.\begin{aligned} p_0(t) &= \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t}, \\ p_1(t) &= 1 - p_0(t) = \frac{\lambda}{\lambda+\mu}(1 - e^{-(\lambda+\mu)t}). \end{aligned}\right\} \quad (7.7)$$

The generalized reliability of system - probability that in torque/moment t it will work:

$$P(t) = p_0(t) = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t}. \quad (7.8)$$

With t $\rightarrow$ - this reliability approaches the limiting value:

$$p = \frac{\mu}{\lambda+\mu},$$

i.e. it is equal to the relative percentage of the intensity of flow of restoration/reductions in the total intensity of flow of restoration/reductions and failures.

Probability $\tilde{P}(t)$ of the fact that to torque/moment t will occur

not one failure, let us determine as follows. Let us assume that there are no restoration/reductions of the refused cell/element, i.e., the graphs of states has the form, shown on Fig. 7.39. The unknown probability $\widetilde{P}(t)$ will be equal to probability $\widetilde{p_0}(t)$ that that system with the graph/count of states, shown on Fig. 7.39, it will be up to torque/moment t in state S_0; this probability will be obtained by the solution of the differential equation

$$\frac{d\tilde{p}_0}{dt} = -\lambda\tilde{p}_0,$$

whence

$$\tilde{p}_0(t) = e^{-\lambda t}.$$

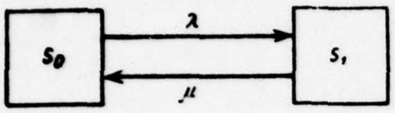

Fig. 7.38.

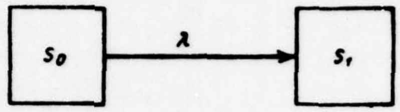

Fig. 7.39.

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Thus,

$$P(t) = e^{-\lambda t}. \quad (7.9)$$

Task 4 (system from several cell/elements with the delayed restoration/reduction).

System S consists of n of the cell/elements each of which is located under the action of the simplest flow of failures with intensity λ. With the failure of any cell/element, the system is disconnect/turned off and begins the restoration/reduction of cell/element. In the inoperative system the cell/elements reject cannot. The intensity of flow of restoration/reductions is equal to μ. All flows - simplest. To find:

- the generalized reliability system P(t) (probability that in torque/moment t system will work);

- Determination the generalized reliability of system p;

- probability $\tilde{P}(t)$ of the fact that to torque/moment t of failures will not at all be.

Solution. System may as before to be only in two states:

S_0 - works,

S_1 is turned off, is restored one cell/element[1].

FOOTNOTE [1]. Simultaneous breakdown of two or more cell/elements is not examined by the force of the ordinariness of the flow of failures. ENDFOOTNOTE.

The graph/count of states is shown on Fig. 7.40. As is evident, it differs from graph in Fig. 7.38 only in terms of the fact that instead of λ stands nλ. Hence, on the basis of the solution of the previous problem,

$$P(t)=\frac{\mu}{n\lambda+\mu}+\frac{n\lambda}{n\lambda+\mu}e^{-(n\lambda+\mu)t}, \tag{7.10}$$

$$p=\frac{\mu}{n\lambda+\mu}, \tag{7.11}$$

$$\tilde{P}(t)=e^{-n\lambda t}. \tag{7.12}$$

Another picture we will obtain, if let us assume that during the restoration/reduction of one cell/element others are continued to work and they can go out of order.

Task 5. System S consists of n of the cell/elements each of which is located under the action of the flow of failures (malfunctions) with intensity λ.

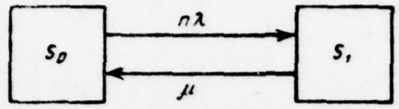

Fig. 7.40.

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With the failure of cell/element, it immediately begins to be restored, remaining cell/elements are continued to work (it is active or in hot reserve). The intensity of flow of the restoration/reductions of the cell/element (independent of the number of simultaneously reducible cell/elements) is equal to μ.

To find:

- probability P(t) of the fact that at torque/moment t all cell/elements will be exact;

- the maximum probability p of the same event;

- the average number of exactly working cell/elements for maximum conditions/mode (with $t \rightarrow \infty$).

Solution. Let us label the states of system according to the number of defective cell/elements:

S_0 - all cell/elements are exact;

S_1 - one cell/element is restored the others are exact;

. . . .

$S_k - k$ of cell/elements are restored the others are exact.

. . . .

S_n - all n of cell/elements are restored.

The graph/count of the states of system is shown on Fig. 7.41. Equate/comparing him with the graph/count of the states locked SMO in the case when ~~Mach~~ m number of workers, that operate machine tools, is equal to number n of the machine tools (see §8 chapters 5), we see that they coincide. Consequently, for both graph/counts coincide differential equations for the probabilities of states, and maximum probabilities. Differential equations take the form:

$$\left.\begin{aligned}
&\frac{dp_0}{dt}=-n\lambda p_0+\mu p_1,\\
&\frac{dp_1}{dt}=-[(n-1)\lambda+\mu]p_1+n\lambda p_0+2\mu p_2,\\
&\dots\dots\dots\dots\dots\dots\dots\\
&\frac{dp_k}{dt}=-[(n-k)\lambda+k\mu]p_k+(n-k+1)\lambda p_{k-1}+(k+1)\mu p_{k+1},\\
&\dots\dots\dots\dots\dots\dots\dots\\
&\frac{dp_n}{dt}=-n\mu p_n+\lambda p_{n-1},
\end{aligned}\right\}\quad(7.13)$$

plus the condition

$$p_1+p_2+\dots+p_n=1.$$

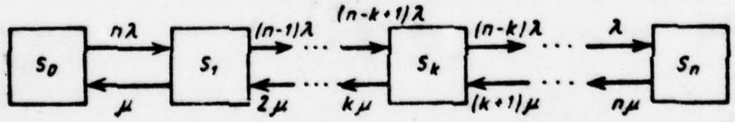

Fig. 7.41.

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The unknown probability $P(t)$ there is nothing else but $p_0(t)$, which we will obtain, integrating system of equations (7.13) under the initial conditions:

$$t=0;\quad p_0=1;\quad p_1=\dots=p_n=0.$$

The maximum probabilities of states we find through the formulas of § 8 of Chapter 5, set/assuming $m = n$, $\lambda/\mu = \rho$:

$$p_0=\frac{1}{1+\frac{n}{1!}\rho+\dots+\frac{n(n-1)\dots(n-r+1)}{r!}\rho^r+\dots+\frac{n(n-1)\dots 2\cdot 1}{n!}\rho^n}=$$

$$=\frac{1}{1+C_n^1\rho+\dots+C_n^r\rho^r+\dots+C_n^n\rho^n}=\frac{1}{(1+\rho)^n};$$

$$p_r=C_n^r\rho^r p_0;\quad r=1,\dots,n.$$

The unknown maximum probability p will be equal to the maximum probability p_0.

The average number of exactly working cell/elements $\bar{u}$ will be equal to the number of cell/elements n, multiplied by probability that the separate cell/element works exactly. This probability for maximum conditions/mode is equal to $\mu/\lambda + \mu$, whence

$$\bar{u}=\frac{n\mu}{\lambda+\mu}=\frac{n}{1+\rho}.$$

The examined problems and examples show that the mathematical apparatus, used for the analysis of the reliability of technical equipment/devices, in essence, coincides with the apparatus of queueing theory, and research of the processes, which take place in systems with unreliable cell/elements, under known conditions can be carried out by the methods of the theory of continuous Markov chains. For this, it is necessary, in order to the flows of events, which translate cell/elements from state into state, they were (it is accurate or approximately) Poisson. These flows not necessarily must be stationary, but in any case similar so that the intensities of flow of events would not depend on the random torque/moments of the transitions of system from state into state. For the simplest, stationary case this means that, in particular, all laws of reliability must be exponential, and the laws of time allocation of restoration/reduction - by also exponential or close to exponential.

8. Account to the dependence of failures during the estimation of the reliability of technical equipment/devices.

Until now, analyzing the reliability of the technical equipment/devices (systems), comprised of cell/elements, we assumed

that the failures of these cell/elements occur independently of each other.

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This assumption not is always correct: in a series of cases of failure, of cell/elements can be dependent.

The dependence between failures can be two types.

1. Failure of any cell/element varies mode of operation of system (for example, can arise short circuit or sharp fluctuations of stress; or breakdown of one cell/element, which is regulator, varies mode of operation of others).

2. On entire set of cell/elements, functions some random factor (temperature, vibration, etc.), which simultaneously affects reliability of all cell/elements or part of them.

Let us pause briefly at the methods of the account of both types of dependence.

Let there be presently the dependence of first type failures - breakdown of one cell/element it affects the operating mode and,

which means, that to the reliability of the others. It is obvious, if we deal with simple (unreserved) system in the absence of restoration/reduction, then first type dependence cannot pronounce on the reliability of system. But if system is reserved (or occurs restoration/reduction), a dependence of such type must be considered.

Example 1. System consists of two cell/elements: basic $Э_1$ and spare $Э_2$, worker of "hot reserve" (Fig. 7.42). With the failure of basic cell/element, the system is automatically switched to spare. The intensity of flow of the failures of both cell/elements in normal working order is identical and equal to λ. Breakdown of basic cell/element affects the mode of operation of spare so that the rate of failures λ increases by value $f(t-t_1)$, where t_1 - torque/moment of the failure of basic cell/element. Thus, the conditional rate of failures of spare cell/element when the basic refused at torque/moment t_1, was equal to:

$$\lambda_2(t/t_1)=\begin{cases}\lambda & (1)\ \text{при}\ t<t_1,\\ \lambda+f(t-t_1) & (1)\ \text{при}\ t>t_1.\end{cases}$$

Key: (1). with.

It is required to determine the reliability of system P(t).

Solution. This problem is reduced to already solved is earlier. It is real/actual, set/assuming $\lambda_1(t) = \lambda_2(t) = \lambda$; $\tilde{\lambda}_2(t/t_1) = \lambda +$

$f(t-t_1)$, we come to that diagram which was examined in problem 1 of §6.

The first type of the dependence of failures (effect of the failures of some cell/elements on the reliability of others) is observed and then, when some cell/elements (regulators) are intended for maintaining the normal mode of the work of others.

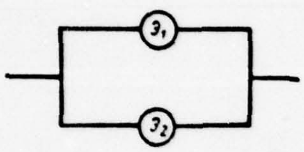

Fig. 7.42.

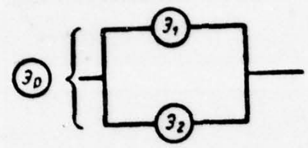

Fig. 7.43.

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Example of 2. System S consists of two "in parallel" connected cell/elements: basic $Э_1$ and spare $Э_2$, that is located in the lightened reserve (Fig. 7.43). Regulator $Э_p$ is intended for supporting the of normal mode of the work of both cell/elements: $Э_1$ and $Э_2$. In the normal mode of the rate of failures of worker and not working (exact) of cell/elements are equal to respectively λ_1 and λ_2. With the failure of regulator, these intensities instantly increase and become equal to $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$. The intensity of flow of the failures of regulator itself is equal to λ_p All flows of events - simplest. To determine the reliability of diagram.

Solution. At constant failure rates the process, which occurs in system - Markov.

Let us label the states of system by three indices: the first is

equal to zero, if is exact regulator, and it is equal to unity, if it left the system. The second index is equal to zero, if is exact basic cell/element Э₁. and to unity, if it left the system. The third index - the same for a spare cell/element Э₂.

The states of system (Fig. 7.44):

S_{000} - all three cell/elements are exact;

S_{010} - regulator is exact, cell/element $Э_1$ left the system, works $Э_2$;

S_{001} - regulator is exact, cell/element $Э_1$ exact, it works; $Э_2$ left the system;

S_{011} - regulator was exact, both cell/element $Э_1$ and $Э_2$ left the system;

S_{100} - regulator left the system; both cell/element $Э_1$ and $Э_2$ were exact, of them $Э_1$ works;

S_{110} - regulator left the system, cell/element $Э_1$ left the system, works $Э_2$;

S_{101} - regulator it left the system, cell/element $Э_1$ works, $Э_2$ failed;

S_{111} - all three cell/elements left the system.

After comprising for this graph/ccunt the system of the differential equations (we let this to do for reader) and after solving these equations under the initial conditions:

$$t=0;\ p_{000}=1;\ p_{.10}=\ldots=p_{111}=0,$$

we will obtain the probabilities of states. The reliability of system P(t) will be expressed as sum of the probabilities of all states, except S_{011} and S_{111}, in which works none of the cell/elements $Э_1$ and $Э_2$:

$$P(t)=1-p_{011}(t)-p_{111}(t). \qquad (8.1)$$

Let us pause now at the second type of the dependence between failures. This type of dependence is caused by the presence of some random factors, which affect simultaneously the work of all cell/elements. Let us consider that these factors determine one or ancther mode of operation of system.

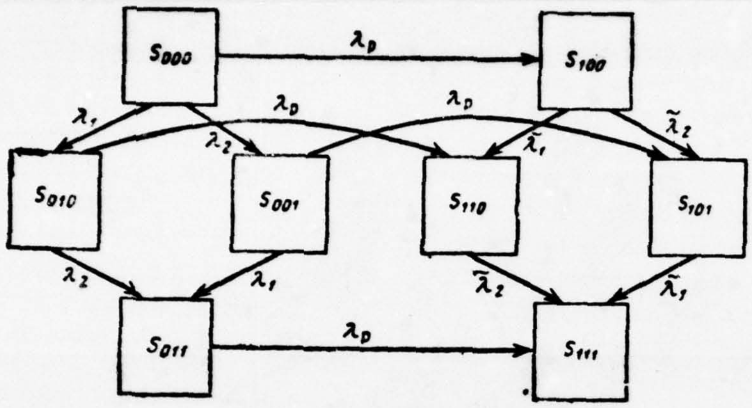

Fig. 7.44.

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Let us consider first the simplest case when the mode of operation of system does not vary in the course of its operation, but it remains constant. Thus, for instance, it is possible to count that the meteorological conditions do not vary or little vary in the process of rocket flight of "Zeml - Zeml's class".

Let be possible several operating modes:

$$R_1, R_2, \ldots, R_h$$

with the probabilities, equal to respectively

$$P(R_1), \quad P(R_2), \ldots, P(R_h).$$

There is certain system S whose reliability depends on the conditions/mode, by which it works. Let us designate the conditional reliability of system during i conditicns/mode (R_i):

$$P(t/R_i) \quad (i=1, \ldots, k).$$

Let us find now the full/total/complete (unconditional) reliability of system P(t). On the formula of the composite probability:

$$P(t)=P(R_1)P(t/R_1)+P(R_2)P(t/R_2)+\ldots+P(R_k)P(t/R_k),$$

or, it is shorter,

$$P(t)=\sum_{i=1}^{k} P(R_i)P(t/R_i). \qquad (8.2)$$

Example 3. System S consists of two "consecutively" connected cell/elements $Э_1$ and $Э_2$ and can work in one of the three conditions/modes: R_1, R_2, R_3, probability of which

$$P(R_1)=0,4;\ P(R_2)=0,3;\ P(R_3)=0,3.$$

During conditions/mode R_1 of the intensity of flow of the failures of elements $Э_1$ and $Э_2$, are equal to 0.1 and 0.2 (failures in hour), during conditions/mode R_2, they are equal to 0.3 and 0.4, during conditions/mode R_3 - 0.4 and 0.5. To determine the reliability

of system and to compute it for t = 2 hour.

Solution. During the "consecutive" connection/compound of the cell/elements of failure rate, they store/add up. We find the conditional reliability of system during three conditions/modes:

$$P(t/R_1)=e^{-(0,1+0,2)t}=e^{-0,3t},$$
$$P(t/R_2)=e^{-(0,3+0,4)t}=e^{-0,7t},$$
$$P(t/R_3)=e^{-(0,4+0,5)t}=e^{-0,9t}.$$

Hence

$$P(t)=0,4\ e^{-0,3t}+0,3e^{-0,7t}+0,3e^{-0,9t}.$$

Set/assuming t = 2, we will obtain:

$$P(2)=0,4\,e^{-0,6}-0,3\,e^{-1,4}+0,3\,e^{-1,8}=$$
$$=0,4\cdot 0,549-0,3\cdot 0,247+0,3\cdot 0,165=0,343.$$

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Of analogous with the examined discrete diagram several conditions/modes it is possible to determine the reliability of system, if the mode of operation is characterized by certain continuous random variable R (let us say, that by temperature), that has known density of distribution f(r). Then in formula (8.2) instead of the sum will figure the integral:

$$P(t)=\int_{(R)} P(t/r)\,f(r)\,dr, \qquad (8.3)$$

where P(t/r) - conditional reliability of system when R = r; f(r) -

the density of distribution of parameter R.

Integral extends to entire range (R) of the possible values of parameter R.

Example 4. System S consists of two cell/elements $Э_1$, $Э_2$, connected "in parallel"; spare cell/element $Э_2$ is located in "hot" reserve. The intensity of flow of the failures of each cell/element is constant in time, but it depends on the mode of operation of system - temperature Θ, this dependence is expressed by the formula

$$\lambda(\Theta)=\lambda_0+\alpha\Theta.$$

The density of distribution of temperature ϑ is constant in range from ϑ_1 to ϑ_2:

$$f(\vartheta)=1/(\vartheta_2-\vartheta_1) \text{ при}^{(1)} \ \vartheta_1<\vartheta<\vartheta_2.$$

Key: (1). with.

To determine the reliability of system.

Solution. We determine the conditional reliability of system at the assigned value $\Theta = \vartheta$:

$$P(t/\vartheta)=1-(1-e^{-(\lambda_0+\alpha\vartheta)t})^2.$$

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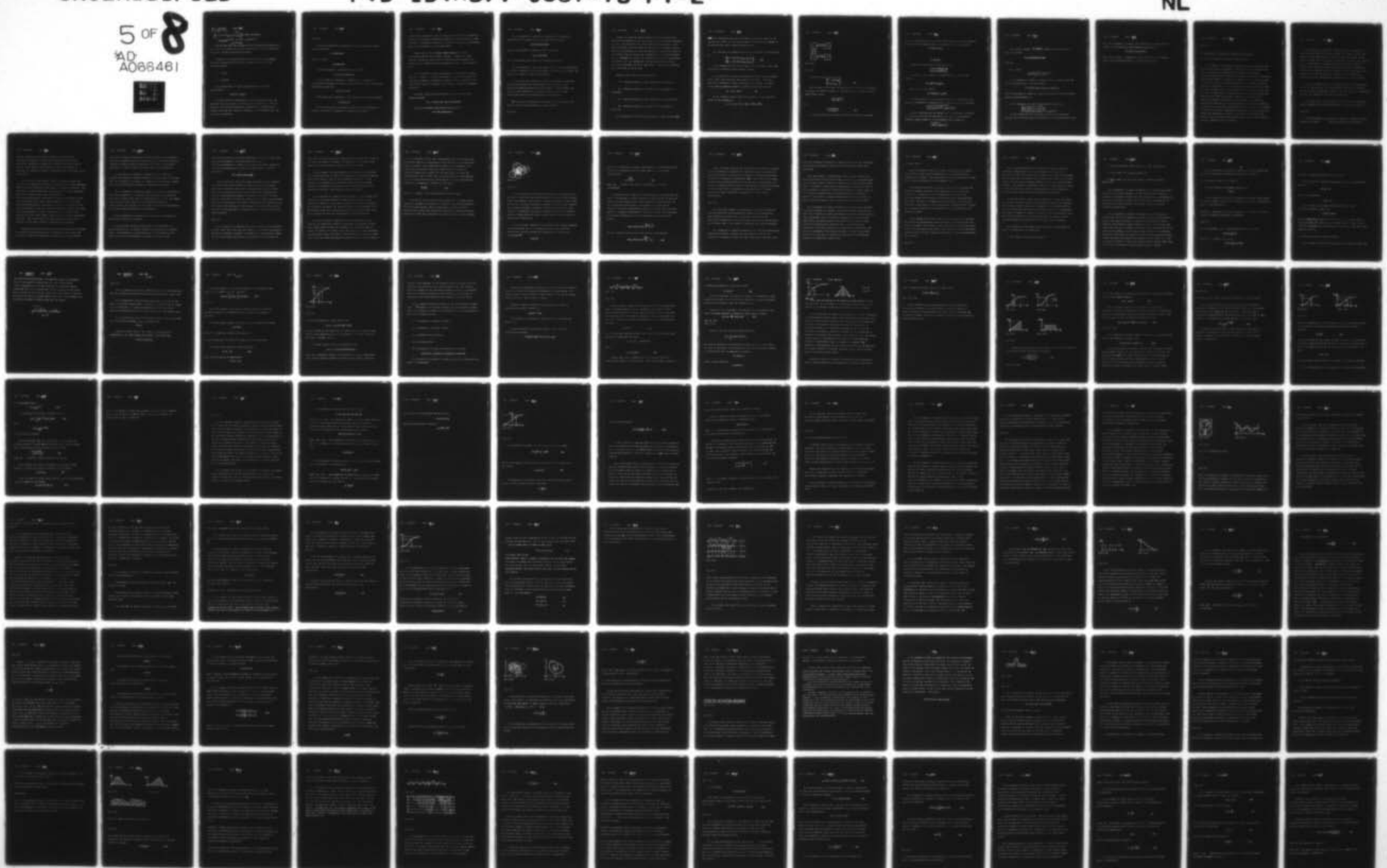

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On formula (8.3)

$$P(t)=$$

$$=\int_{\vartheta_1}^{\vartheta_2}[1-(1-e^{-(\lambda_0+\alpha\vartheta)t})^2]\frac{1}{\vartheta_2-\vartheta_1}\,d\vartheta=\frac{1}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\vartheta_2}[2e^{-(\lambda_0+\alpha\vartheta)t}-e^{-2(\lambda_0+\alpha\vartheta)t}]\,d\vartheta=$$

$$=\frac{1}{\alpha(\vartheta_2-\vartheta_1)t}\Big[2e^{-(\lambda_0+\alpha\vartheta_1)t}-2e^{-(\lambda_0+\alpha\vartheta_2)t}-\frac{1}{2}e^{-2(\lambda_0+\alpha\vartheta_1)t}+$$

$$+\frac{1}{2}e^{-2(\lambda_0+\alpha\vartheta_2)t}\Big].$$

Let us note that the disregard of the dependence of failures, if it is and it is essential, it can lead to large errors, especially, if system consists of many cell/elements.

Example of 5. System **S** consists of 50 uniform cell/elements, connected "consecutively", and it can work in one of the two conditions/modes:

R_1 - normal,

R_2 - abnormal.

The probabilities of these conditions/modes are equal respectively:

$$P(R_1)=0,9;\quad P(R_2)=0,1.$$

In normal mode the reliability of each cell/element (for the specific time τ) is equal p = 0.998, in the abnormal p' = 0.9. To determine the full/total/complete reliability of system **S** and to compare from that that would be obtained, if cell/elements went out of order independently.

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Solution. Conditional reliability of system during the first conditions/mode:

$$P_{K_1} = 0,998^{50} \approx 0,904;$$

with the second

$$P_{K_2} = 0,9^{50} \approx 0,004.$$

Full/total/complete reliability of the system:

$$P \approx 0,9 \cdot 0,904 + 0,1 \cdot 0,004 = 0,814.$$

Let us count TU reliability, considering the failures of cell/elements independent variables and assigning to each of them the reliability, equal to

$$\grave{p} = 0,9p + 0,1p' = 0,988.$$

Multiplying the reliability of 50 cell/elements, we will obtain:

$$\tilde{p} = 0,988^{50} \approx 0,551.$$

As can be seen from an example, neglect of the dependence of failures during the "consecutive" connection/compound of

cell/elements can lead to the essential understating of reliability. During the "parallel" connection/compound of cell/elements, the same disregard of dependence leads not to the understating of reliability, but, on the contrary, to its overestimate.

Example of 6. The redundant system consists of basic cell/element $Э_1$ and three spare: $Э_2, Э_3, Э_4$, working in "hot" reserve. System can work in one of the two conditions/modes: R_1 - normal and R_2 - abnormal with probabilities $P(R_1) = 0.7$ and $P(R_2) = 0.3$.

The reliability of all cell/elements is identical; in normal mode it is equal $p = 0.99$, in abnormal $p' = 0.4$. To determine the full/total/complete reliability of system P and to compare by its that $\tilde{P}$ which will be obtained, if we consider failures independent variables.

Solution. Conditional reliability of system during each conditions/mode:

$$P_{R_1} = 1-(1-0{,}99)^4 \approx 1{,}000, \quad P_{R_2} = 1-(1-0{,}4)^4 \approx 0{,}870.$$

Full/total/complete reliability of the system:

$$P = 0{,}7\cdot 1{,}000 + 0{,}3\cdot 0{,}870 = 0{,}961.$$

If we consider the failures of cell/elements independent variables and to ascribe to each of them the reliability

$$\tilde{p} = 0,7 \cdot 0,99 + 0,3 \cdot 0,4 = 0,813,$$

then the reliability of system it will be another:

$$\tilde{P} = 1 - (1 - \tilde{p})^4 \approx 0,999,$$

i.e. considerably higher than true reliability 0.961.

The overestimate of the reliability of the reserved block which is obtained neglecting of the dependence of failures, is greater, the greater the number of spare cell/elements.

If technical system consists of the cell/elements, connected both "consecutively" and "in parallel" (for example, if are duplicate/backup/reinforced only most important node/units), then neglect of the dependence of failures can lead both to the overestimate and to the understating of reliability.

Finally, let us consider the case when in the process of the work of system conditions/mode can vary randomly.

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Example 7. System S, which consists of two "consecutively" connected cell/elements, can work in one of the two conditions/modes: R_1 and R_2. The transition of system from conditions/mode R_1 into conditions/mode R_2 occurs under the action of the simplest flow of events with intensity λ_{12}; reverse transition - under the action of the simplest flow of events with intensity λ_{21}. In conditions/mode R_1, the intensity of flow of the failures of the first cell/element is equal to $\lambda_1^{(1)}$, the second $\lambda_2^{(1)}$; in conditions/mode R_2 these intensities are equal to $\lambda_1^{(2)}$, $\lambda_2^{(2)}$. All flows - simplest. To determine the reliability of system $P(t)$.

Solution. The states of system will be:

$S_{1\text{и}}$ - conditions/mode R_1, both cell/element are exact;

$S_{1\text{н}}$ - conditions/mode R_1, at least one cell/element is defective;

$S_{2\text{и}}$ - conditions/mode R_2, both cell/element are operative;

$S_{2\text{н}}$ - conditions/mode R_2, at least one cell/element is defective.

The graph/count of the states of system is shown on Fig. 7.45.

[redacted] The arrow/pointers, which translate system from state $S_{1н}$ in $S_{2н}$ and vice versa, are not shown, since, it system is not exact, to us nevertheless, which conditions/mode occurs.

The equations of Kolmogorov for the probabilities of states will be:

$$\left.\begin{aligned}\frac{dp_{1н}}{dt}&=-\left(\lambda_1^{(1)}+\lambda_2^{(1)}+\lambda_{12}\right)p_{1н}+\lambda_{21}\,p_{2н},\\ \frac{dp_{2н}}{dt}&=-\left(\lambda_1^{(2)}+\lambda_2^{(2)}+\lambda_{21}\right)p_{2н}+\lambda_{12}\,p_{1н}.\end{aligned}\right\}\quad(8.4)$$

Other probabilities us in this case do not interest, since they correspond to defective (not working) system.

If we know that in which conditions/mode (R_1 or R_2) the system begins work, then equations (8.4) will be integrated under the completely determined initial conditions. for example, if system begins work in conditions/mode R_1, initial conditions will be:

$$t=0,\quad p_{1н}=1,\quad p_{2н}=0.\qquad(8.5)$$

Let us integrate system (8.4), for example, at the numerical values of the parameters:

$$\lambda_{12}=1,\ \lambda_{21}=3,\ \lambda_1^{(1)}=1,\ \lambda_2^{(1)}=2,\ \lambda_1^{(2)}=2,\ \lambda_2^{(2)}=4.$$

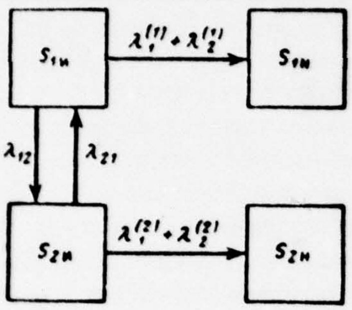

Fig. 7.45.

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Equations (8.4) take the form:

$$\left.\begin{aligned} \frac{dp_{1н}}{dt} &= -4p_{1н}+3p_{2н}, \\ \frac{dp_{2н}}{dt} &= p_{1н}-9p_{2н}. \end{aligned}\right\} \quad (8.6)$$

Let us first of all find that with which λ the pair of functions $Ce^{-\lambda t}$, $De^{-\lambda t}$ can satisfy equations. The substitution of this pair into system (8.6) gives:

$$\left.\begin{aligned} -\lambda C &= -4C+3D, \\ -\lambda D &= C-9D \end{aligned}\right\}$$

or

$$\left.\begin{aligned} (\lambda-4)C+3D &= 0, \\ C+(\lambda-9)D &= 0. \end{aligned}\right\} \quad (8.7)$$

So that the system of equations (8.7) would have any solution

(C, D), besides zero, it is necessary and sufficient so that would be equal to zero determinant from the coefficients of this system:

$$(\lambda-4)(\lambda-9)-1\cdot 3=0$$

or

$$\lambda^2-13\lambda+33=0.$$

Solving this equation, we find two values λ:

$$\lambda_1=6,5-\sqrt{6,5^2-33}\approx 3,459,$$
$$\lambda_2=6,5+\sqrt{6,5^2-33}\approx 9,541.$$

At value $\lambda=\lambda_1$, solution of system (8.7) is given by the formula

$$D^{(1)}=\frac{4-\lambda_1}{3}C^{(1)}=0,180\,C^{(1)},$$

when $\lambda=\lambda_2$ - by formula

$$D^{(2)}=\frac{4-\lambda_2}{3}C^{(2)}=-1,847\,C^{(2)}.$$

It hence follows that the general view of the solution of the system of differential equations (8.7) - this pair of the functions:

$$p_{1n}(t)=C^{(1)}e^{-3,459t}+C^{(2)}e^{-9,541t},$$
$$p_{2n}(t)=0,180\,C^{(1)}e^{-3,459t}-1,847\,C^{(2)}e^{-9,541t}.$$

Initial conditions we can satisfy by the appropriate selection of arbitrary constants $C^{(1)}$ and $C^{(2)}$. So that would be implemented condition $p_{1n}(0)=1$, $p_{2n}(0)=0$, it is necessary that it would be

$$\left.\begin{aligned}&C^{(1)}+C^{(2)}=1,\\&0,180\,C^{(1)}-1,847\,C^{(2)}=0.\end{aligned}\right\}$$

From second equation $C^{(2)}=0{,}097C^{(1)}$; substituting this in the first, we obtain:

$$C^{(1)}=1/1{,}097=0{,}912;\ C^{(2)}=0{,}088.$$

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It is final:

$$p_{1н}(t)=0{,}912\,e^{-3{,}459\,t}+0{,}088\,e^{-9{,}541\,t},$$
$$p_{2н}(t)=0{,}164\,(e^{-3{,}459\,t}-e^{-9{,}541\,t}).$$

The reliability of system, obviously, will be equal to the sum of the probabilities of the exact work:

$$P^{(1)}(t)=p_{1н}^{(1)}(t)+p_{2н}^{(1)}(t)=1{,}076\,e^{-3{,}459\,t}-0{,}076\,e^{-9{,}541\,t},$$

where superscript (1) shows that they are calculated for the specific initial conditions/mode R_1.

It is analogous, for the initial conditions/mode R_2:

$$p_{1н}^{(2)}(t)=0{,}493\,(e^{-3{,}459\,t}-e^{-9{,}541\,t}),$$
$$p_{2н}^{(2)}(t)=0{,}089\,e^{-3{,}459\,t}+0{,}911\,e^{-9{,}541\,t},$$
$$P^{(2)}(t)=p_{1н}^{(2)}(t)+p_{2н}^{(2)}(t)=0{,}582\,e^{-3{,}459\,t}+0{,}418\,e^{-9{,}541\,t}.$$

If the initial mode of operation of system in accuracy is unknown, and are known only to probability of conditions/modes R_1 and

R_2 in the beginning of process, the reliability of system can be counted on the formula of the composite probability:

$$P(t)=P(R_1)P^{(1)}(t)+P(R_2)P^{(2)}(t),$$

where $P(R_1)$, $P(R_2)$ - probability of the fact that at the initial moment occur conditions/modes R_1 and R_2 respectively.

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8. SIMULATION OF OPERATIONS BY STATISTICAL TESTINGS.

1. Method of statistical testings (Monte-Carlo).

In a number of the previous chapters we became acquainted with the methods of the construction of some mathematical models, which give possibility to establish/install analytical (formula) communication/connection between the assigned conditions of operation (among other things by accepted us solution) and the result (issue) of operation, that are characterized by one or several parameters - indices of efficiency. If in the course of operation there are mixed in random factors, then it represents by itself random process, and the index of efficiency - probability of some event or mathematical expectation of some random value. Sometimes to construct the analytical model of random process (for example, the system of differential equations for the probabilities of states or algebraic equations for the maximum probabilities of states) and it is possible to connect the assigned conditions of operation with its issue with analytical dependences. However, this accomplishes hardly ever, mainly, when the random process, which takes place in the system in question, Markov or close to Markov.

In practice by no means all random processes, observed in operations, are Markov or close to them. For example, in the real systems of mass maintenance the flow of claims in any way is not always Poisson; even more rarely is observed exponential (or close to it) time allocation of maintenance. For the arbitrary flows of events, which translate system from state into state, analytical solutions are obtained only for isolated special cases, but in the general case of the satisfactory methods of the mathematical description of the corresponding processes, there does not exist.

When the construction of the analytical model of phenomenon on one or the other reason is is difficultly realizable, is applied another method of simulation, known by the name method of statistical testings or, otherwise, the Monte-Carlo method.

At present during the simulation of operations and generally random processes, the Monte-Carlo method is applied very widely.

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This wide acceptance of method is connected, mainly, with the appearance of ETsVM [digital computer], which make it

possible within the foreseeable periods to implement mass calculations according to this method (without machines very laborious). However, in principle the Monte-Carlo method can be applied, also, without aid of ETsVM. In this paragraph we is presented the essence of method, irrespectively of the method of its realization.

The idea of the Monte-Carlo method is extremely simple and consists it of following. Instead of, describing of random phenomenon with the help of analytical dependences, is produced "drawing" - the simulation of random phenomenon with the help of certain procedure, which gives random result. Just as in life the concrete/specific/actual realization of process store/adds up each time differently, that gives random result. Just as in life the concrete/specific/actual realization of process store/adds up each time differently, so and as a result of "drawing" we obtain one copy - one "realization" of random phenomenon. After producing this "drawing" the very large number once, we will obtain the statistical material - many realizations of random phenomenon, which can be developed by the usual methods of mathematical statistics.

Frequently this method proves to be more simply than the attempt to construct the analytical model of phenomenon and to trace the dependence between its parameters on this model. For the complex

operations in which participates the large number of cell/elements (machines, systems, people, collectives) and in whom random factors complexly interact between themselves, the method of statistical testings, as a rule, proves to be more simple than analytical.

In essence, by the method of "drawing" can be solved any probabilistic problem; however justified it becomes only in the case when the procedure of "drawing" is simpler, but are not more complex than the application/use analytical, computational methods.

Let us consider an elementary example. Is solved the problem: on some target/purpose Ts, is produced four independent variables of shot each of which falls into it with probability p = 0.5. For damage (destruction) to the target of one incidence/impingement, insufficiently, is required not less than two incidence/impingements. To determine kill probability to target/purpose.

Stated probabilistic problem can be solved by two methods: a) analytically and by b) "drawing".

Is first solved problem analytically. Kill probability to target/purpose W let us compute through the probability of opposite event - nondestruction of target/purpose. The probability of the nondestruction of target/purpose is equal to the sum of the

probabilities not of one incidence/impingement and it is exact one incidence/impingement; the probability not of one incidence/impingement is equal to 0.5^4; the probability exactly of one incidence/impingement is equal to $C_4^1 \cdot 0.5^1 \cdot 0.5^3 - 4 \cdot 0.5^4$, therefore,

$$W = 1 - (0{,}5^4 + 4 \cdot 0{,}5^4) \approx 0{,}688.$$

Now let us try to solve TU problem "drawing". Let us simulate the procedure of shooting with the help of other, also random, procedures. Instead of four shots on target/purpose, let us throw four coins: the appearance of a coat of arms will conditionally indicate "incidence/impingement", and tail - "error". If of four deserted coins not less than two fall by coat of arms, this will mean that the target/purpose "is struck". "experiment" or "drawing" in our case will be the throwing of four coins; by "result" either by the "issue" of this experiment - "damage/defeat" or "nondestruction" of target/purpose.

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Let us repeat this "experiment", which consists of the throwing of four coins, very there are many times in a row. Then, according to Bernoulli's theorem, frequency of "damage" the target/purpose almost for sure will differ little from the probability of this event W;

that means if we is thrown four coins the large number once N and let us divide the number of "damage" to target into N, we almost will obtain for sure the number, close to W, i.e., to 0.688.

In this example the determination of probability W by drawing was incomparably more difficult than analytical calculation. However, hardly ever this is thus. It very frequently proves to be that obtaining the probability of event (or average value of random variable) by analytical, calculation is so complicated and it is cumbersome that more simply proves to be the drawing.

Let us consider an example of this problem. Let be produced the bombing on some target/purpose Ts (Fig. 8.1); the zone of the destructive effect of bomb takes the form of the circle of radius r. Is discarded n of bombs. For damage to target (her breakdown) it is necessary to cover with destruction not less than ko/o target area. It is required to find kill probability to target/purpose W.

In spite of visible simplicity of the formulation of the problem, its analytical solution is extremely complicated. It will much simpler solve problem by "drawing". For this, will have to "play" coordinate n of impact points (as this to do - will described subsequently); around each impact point describe the circle of radius r and count the common/general/total affected target area (in Fig. 8.1

it is shaded). If this area render/showed more ko/o target area, to consider that the target/purpose "was struck", if less than ko/o - "was not struck". This "experiment", which is of "throwing" n of bombs, must be repeated very many times, noting each time by arbitrary symbol (for example "+") the experiment, in which the target/purpose was "struck". With the large number of "experiments" in N, the kill probability to target/purpose W can be approximately estimated as frequency of "damage" to the target:

$$W \approx \frac{M}{N}, \qquad (1.1).$$

where M - a number of the "experiments", noted by plus.

It proves to be that even for the examined by us comparatively elementary problem the procedure of "drawing" (obtaining probability the Monte-Carlo method) will be considerably simpler than the determination of the same probability analytical, calculated method. Example is a good specimen/sample of typically "Monte Carlo" problem.

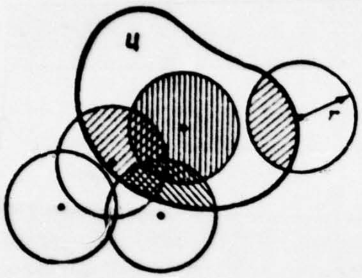

Fig. 8.1.

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Let us note that by statistical testings (Monte Carlo) it is possible to find not only probabilities of events, but also average value (mathematical expectations of random variables. In this case, we will use no longer Bernoulli's theorem, but law of large numbers (by Chebyshev theorem). According to this theorem with the large number of independent experiments arithmetic mean of the observed values of random quantity almost differs little for sure from its mathematical expectation.

So, if we under conditions of the last/latter example (bombing on target/purpose Ts) it is required for us to find not kill probability, but the mathematical expectation of the area of destruction S_p:

$$M[S_p] = \bar{s}_p,$$

then it it is possible to define approximately as arithmetic mean of the areas of destruction in the large number N of the played "experiments":

$$\bar{s}_p \approx \frac{\sum_{i=1}^{N} S_{pi}}{N}, \qquad (1.2)$$

where S_{pi} - a value of the area of destruction in the i-th "experiment".

Analogously can be found not only mathematical expectations, but also the dispersions of which interest us random values. We will not forget, that the dispersion of random variable is nothing else but the mathematical expectation of the square of central random variable; it can be approximately found as arithmetic mean of these squares in separate "experiments". So, in our example the dispersion of the area of destruction can be approximately found from the formula

$$D[S_p] = M\left[(S_p - \bar{s}_p)^2\right] \approx \frac{\sum_{i=1}^{N} (S_{pi} - \bar{s}_p)^2}{N}$$

or, that simpler, through the second initial torque/moment:

$$D[S_p] = M[S_p^2] - \bar{s}_p^2 \approx \frac{\sum_{i=1}^{N} S_{pi}^2}{N} - \bar{s}_p^2. \qquad (1.3)$$

Thus, the Monte-Carlo method in operations research is a method of the mathematical simulation of random phenomena, in which very chance directly is included in the process of simulation represents by itself its essential cell/element. Each time, when in the course of operation is add/interfered one or the other random factor, its effect is imitated with the help of the specially organized "drawing" or "coin toss". Thus is constructed one realization of random phenomenon, which represents by itself as if result of one "experiment".

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With the large number of realizations, the which interest us characteristics of random phenomenon (probability, mathematical expectations) are located so, as they are located from the experiment: probability - as frequencies, mathematical expectations - as arithmetic mean of the values of corresponding random variables.

The simulation of random phenomena by the Monte-Carlo method has common/general/total features with the process of the set of experiment by separate people and human collectives. And here, and

that each separate realization is randcm; stable laws are discovered cnly during the repeated observation of phenomenon, during vast experiment.

The large number of realizations, which requires during the application/use of Monte-Carlo method, is made it by generally bulky and laborious. Before starting the Monte-Carlo method, always makes sense to attempt to solve problem analytically, and only if this it were possible, to resort to statistical simulation. Useful proves to be at least the approximate preliminary analytical solution of problem - this helps to reveal/detect/expose the basic factors on which depends the result, and to plan the plan cf further work.

The simulation of random phenomena by the Monte-Carlo method is frequently produced for purpose of checking competence/validity in this case of one or the other mathematical apparatus, always based on some assumptions. Let, for example, examining the system of mass maintenance, we approximately replace ncn-Poisson flow of claims Poisson and nonindicative servicing time - exponential. The simulation of the same process by the Mcnte-Carlo method will show that are permissible these simplifications, to which errors they lead, and it will make it possible tc introduce into calculation formulas the appropriate corrections.

2. Single toss.

Basic cell/element from set of which store/adds up the Monte Carlo model, is one random realization of the phenomenon being simulated, for example: one "bombardment" of target/purpose, one "day of the work" of transport, one "epidemic", etc.

Realization represents by itself as one case of realizing the random phenomenon (process) being simulated with all which are inherent in it chances. It is developed with the help of the specially worked out procedure or the algorithm in which important role plays strictly the "drawing" or the "throwing of toss". Each time when in the course of the process being simulated there is mixed in a chance, its effect is considered not by calculation, but the throwing of toss.

Let us assume that in the course of the process being simulated did appear the torque/moment when its further development (but that means and result) does depend on that, did appear in this stage event A or it did not appear? For example: did occur the hit? Is discovered certain object? Is exact equipment? Did appear claim for maintenance? and so forth.

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Then it is necessary by the "throwing of toss" to solve the question: appeared event A or did not appear? For this, it is necessary to give into action certain random mechanism of the drawing (let us say, that to throw the die, either several coins or to select number from the table of random numbers) and to agree against which result of toss indicates appearance, and which-nonappearance of event A. Below we will see, that the drawing always can be organized so that the event A would have any preassigned probability.

Besides the events, which appear randomly, to course and issue of operation can also affect various random variables (for instance, the servicing time of claim by the channel SMO; coordinates of the impact point of projectile; the time, during which it is implemented the voyage of truck; the number of left the system node/units, etc.). With the help of toss it is possible to play the value of any of random variable or the value part of several random variables.

let us agree to call single toss any elementary experiment in which is solved one of the questions:

1. Did occur or did not occur event A?

2. Which of possible events $A_1, A_2, \ldots, A_k$ did occur?

3. Which value took random variable X?.

4. Which value part did take system of random variables $X_1, X_2, \ldots, X_k$?

Each realization of random phenomenon by the Monte-Carlo method consists of the chain/network of single tosses, which interrupt themselves by routine calculations. By calculations is considered the effect of the issue of single toss on the course of operation (in particular, to the conditicrs under which it will be realized following single toss).

Let us consider the methods of organizing all varieties of single toss. As it was already said abcve, during any organization of single toss, must be started to course some mechanism of random sampling (coin-tossing, bones, remcval cf badge from rotating drum, number from the set of numbers and, etc.). Such mechanisms can be the most diverse; however, any cf them can be replaced by the standard mechanism, which makes it possible tc solve one-only problem: obtain random variable, distributed with constant density from 0 to 1. Let us agree for brevity to call this randcm variable "random number from 0 to 1" and to designate R (from English random - random).

Let us show that any problem of single toss can be solved with the help of the standard mechanism, which gives number R.

1. Did appear or there is no event A?

Let the probability of event A be equal p:

$$P(A)=p.$$

Let us select with the help of standard mechanism random number R and will consider that if it is lesser than p, event A occurred, if more p - did not occur [1].

FOOTNOTE [1]. Obtainings R, in the accuracy of the equal to p, let us consider virtually impossible. ENDFOOTNOTE.

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It is real/actual, if R - random number from 0 to 1, then

$$P(R<p)=\int_0^p f(r)\,dr,$$

where (r) = 1 with 0 < r < 1, or

$$P(R<p)=\int_0^p dr=p=P(A).$$

2. Which of several possible events did appear?

Let there be the full/total/complete group of the antithetical events:

$$A_1, A_2, \ldots, A_k$$

with the probabilities

$$p_1, p_2, \ldots, p_k.$$

Since events are incompatible/inconsistent and form full/total/complete group, then

$$p_1+p_2+\ldots+p_k=1.$$

Let us divide entire range from 0 to 1 into k of sections with a length of $p_1, p_2, \ldots, p_k$ (Fig. 8.2). If random number R, given out by standard mechanism, hit, for example, to section p_3, this means that appeared event A_3.

3. Which value did take random variable?

Let we need "to play" the value of random variable X, which has

the known law of distribution. The case when value X is discrete (i.e. it has separate values $x_1, x_2, \ldots, x_k$ with probabilities $p_1, p_2, \ldots, p_k$) examine we will not, since it is reduced to previous point/item by 2. It is real/actual, if we designate A_i the event, which consists of the fact that value X did take value x_i, then the drawing of the value of random value X is reduced to the solution of the question: which of events $A_1, A_2, \ldots, A_k$ did appear?

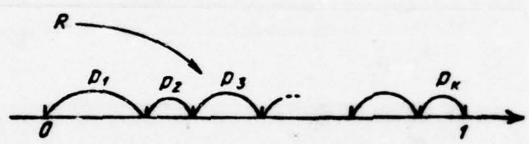

Fig. 8.2.

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Let us examine the case when random variable X is continuous and has the assigned continuous function of distribution F(x) (Fig. 8.3).

Let us demonstrate the following assertion: if we take on the axis of ordinates random number R (from 0 to 1) and to find the value of X at which F(X) = R (see arrow/pointers in Fig. 8.3), then obtained random variable X let us have function of distribution F(x).

It is real/actual, let us take random variable X and let us find its distribution function, i.e., the probability

$$P(X<x).$$

Figures 8.3 shows that so that would be implemented the inequality X < x, value R must take value is less than F(x):

$$P(X<x)=P(R<F(x)).$$

But random number R has constant density of distribution f(r), equal to 1 on segment (0, 1); that means

$$P(X<x)=\int_{0}^{F(x)} f(r)\,dr=\int_{0}^{F(x)} 1\cdot dr=F(x). \qquad (2.1)$$

Q. E. D.

Thus, the drawing of the value of random variable X with the assigned function of distribution F(x) is reduced to following procedure.

To obtain random number R from 0 to 1 and as value of X to take:

$$X=F^{-1}(R),$$

where F^{-1} - function, reverse with respect to F.

4. Which value part will take the system of random variables?

Let there be the system of random variables:

$$X_1,\ X_2,\ \ldots,\ X_n \qquad (2.2)$$

with combined density of distribution

$$f(x_1,\ x_2,\ \ldots,\ x_n).$$

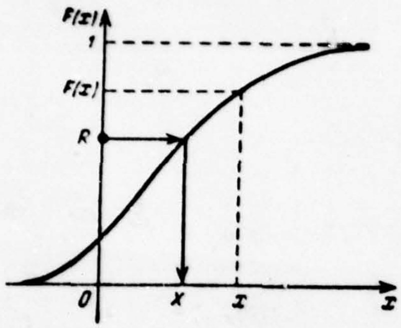

Fig. 8.3.

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If random variables are independent, then

$$f(x_1, x_2, \ldots, x_n) = f_1(x_1) f_2(x_2) \ldots f_n(x_n),$$

and the drawing of the value part of system (2.2) is reduced to play each of them individually, i.e., to organize n of single tosses of the type, described in p. 3.

If random variables (2.2) are dependent, then

$$f(x_1, x_2, \ldots, x_n) = f_1(x_1) f(x_2/x_1) f(x_3/x_1 x_2) \ldots,$$

where each subsequent density of distribution is taken conditional, when previous random variables took the specific values. With the

drawing of the sequence of the values of random variables (2.2) is obtained first the value x_1 of random variable X_1; this value is taken as argument in the conditional density $f(x_2/x_1)$; is developed value x_2 of random variable X_2, both values x_1, x_2 are taken as arguments in the conditional density $f(x_3/x_1, x_2)$ and so forth.

Let us consider several examples for the organization of single toss. Example 1. The flight vehicle, which accomplishes flight above the territory of enemy, after shooting at it can render/show in one of the following states:

A_1 is unharmed, is continued flight;

A_2 is injured, is continued flight;

A_3 - completed forced landing;

A_4 was biased/beaten.

The probabilities of these four events are assigned:

$$p_1=P(A_1)=0.4;\quad p_2=P(A_2)=0.1;\quad p_3=P(A_3)=0.15;\quad p_4=P(A_4)=0.35.$$

To construct the procedure of single toss for the drawing of the result of bombardment.

Sclution. Is divisible section (0, 1) cn four parts, as shown in Fig. to 8.4. With the incidence/impingement of random number R to section from 0 to 0.4 to consider that occurred event A_1, to section from 0.4 to 0.5 - event A_2 and so forth.

Example 2. Random variable X is distributed according to exponential law with a density of:

$$f(x)=\lambda e^{-\lambda x} \quad (x>0).$$

To construct the procedure of single toss for obtaining the value of X.

Sclution. Through the assigned density f(x) we find the distribution function:

$$F(x)=\int_0^x f(x)\,dx=\int_0^x \lambda e^{-\lambda x}\,dx = 1-e^{-\lambda x} \quad (x>0).$$

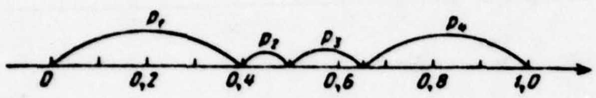

Fig. 8.4.

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Plotted function F(x) is given in Fig. to 8.5. Graphically the value of random variable X it is possible to play thus: to take random number from 0 to 1 on the axis of ordinates and to find that corresponding to it the value of abscissa X (see arrow/pointers in Fig. 8.5). This can be done not graphically, but by calculation, if we write:

$$R = 1 - e^{-\lambda X} \tag{2.3}$$

and to solve this equation relative to X (i.e. to find inverse with respect to F function). We have:

$$e^{-\lambda X} = 1 - R; \qquad -\lambda X = \ln(1 - R),$$

whence

$$X = -\frac{1}{\lambda}\ln(1 - R). \tag{2.4}$$

Formula (2.4) can be simplified; let us recall that if R - random number from 0 to 1, then (1-R) - also random number from 0 to

1; therefore instead of (2.4) can be taken

$$X = -\frac{1}{\lambda} \ln R. \qquad (2.5)$$

Thus, the procedure of drawing is reduced to following: to take random number from 0 to 1, to take the logarithm it with the natural foundation for changing sign and dividing on λ.

Example of 3. To construct the procedure of the drawing of the value of random variable X, density of distribution of which

$$f(x) = {}^1/_2 \cos x \text{ (1) при } -\pi/2 < x < \pi/2 \qquad (2.6)$$

Key: (1). with

(Fig. 8.6).

Solution. We find the distribution function:

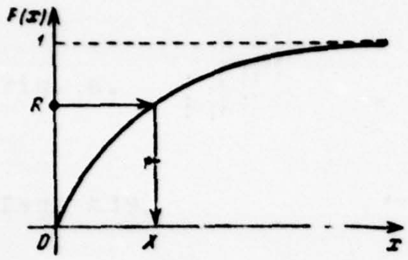
$$F(x) = \int_{-\frac{\pi}{2}}^{x} {}^1/_2 \cos x\,dx = {}^1/_2 (\sin x + 1).$$

The plotted function of distribution is given in Fig. to 8.7. There is shown the procedure of the drawing of the value of random variable X. Analytically this is expressed as follows:

$$R = {}^1/_2 (\sin X + 1),$$

whence inverse function

$$X = \arcsin (2R - 1).$$



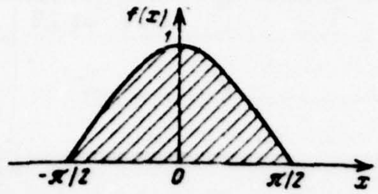


Fig. 8.5

Fig. 8.6

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Thus, for the drawing cf the value of random variable X with density (2.6) it is necessary: to take random number from 0 to 1, double it, to subtract unity and from result tc take arc sine.

Let us note that in examined by us examples of 2 and 3 functions of distribution F random values easily allcw/assumed obtaining in an explicit form of inverse function F^{-1}; in practice this hardly ever is thus. If we explicit expression for inverse function it is impossible obtain, it is pcssible, as shown in Fig. to 8.3, to determine this inverse function by a graph; but if calculation is produced not by hand, but by ETsVM, it is possible to use the method, proposed by N. P. Buslenkc [15]; it lies in the fact that the function of distribution F(x) is replaced by function $F^*(x)$, comprised of the segments of lines (Fig. 8.8); this can be done with any assigned degree of accuracy. For each of such linear sections inverse function is found easily.

Example 4. There is a system of dependent random variables X_1 and X_2. Random variable X_1 is distributed according to the law of

right triangle on section from 0 to 1 (Fig. 8.9):

$$f_1(x_1)=2x_1 \quad \text{при}^{(1)} \quad 0<x_1<1.$$

Key: (1). with.

Random variable X_2 is distributed with constant density on section with a length of 2, with center at point x_1, where x_1 - value, accepted by random variable X_1 (Fig. 8.10). To organize the procedure of single toss for the drawing of the pair of the values of random values X_1, X_2.

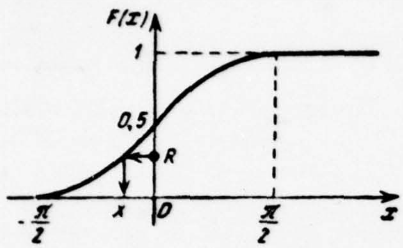

Fig. 8.7.

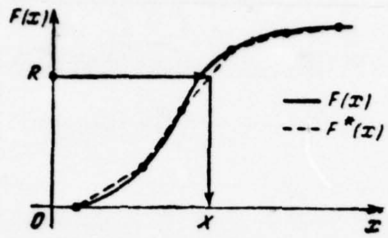

Fig. 8.8.

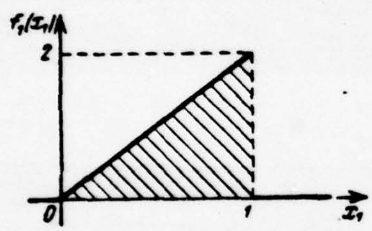

Fig. 8.9.

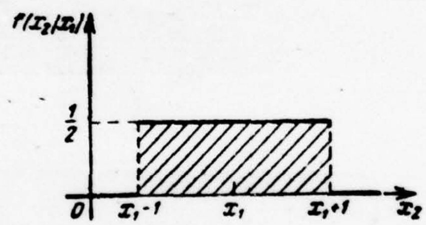

Fig. 8.10.

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Solution. Let us play first the value of value X_1; for this let us construct its the distribution function:

$$F_1(x_1) = \begin{cases} 0 \ \text{при}^{(1)} \ x_1 < 0, \\ x_1^2 \ \text{при} \ 0 < x_1 < 1, \\ 1 \ \text{при} \ x_1 > 1. \end{cases} \quad (2.7)$$

Key: (1). with

(Fig. 8.11). Value X_1 we will obtain as inverse function with respect to (2.7) from random number R:

$$R = X_1^2; \quad X_1 = \sqrt{R}. \tag{2.8}$$

After is played value X_1, it no longer is random; let us designate it x_1. At the known value x_1, we construct the conditional function of distribution $F(x_2/x_1)$ of random variable X_2 (Fig. 8.12). The expression of this distribution function will be:

$$F(x_2/x_1) = {}^1/_2 (x_2 - x_1 + 1) \text{ при}^{(1)} \; x_1 - 1 < x_2 < x_1 + 1. \tag{2.9}$$

Key: (1). with.

Let us take the new random number R' from 0 to 1 and let us find from it the function, reverse (2.9):

$$R' = {}^1/_2 (X_2 - x_1 + 1), \quad X_2 = 2R' + x_1 - 1. \tag{2.10}$$

Thus, the procedure of drawing is reduced to following: is taken random number R from 0 to 1 and from it is extracted the root; the obtained value $\sqrt{R}$ is the played value of first random variable $X_1 = x_1$. Is further taken still one random number R from 0 to 1, it doubles, to it is adjoined that previously obtained x_1 and is subtracted unity; is obtained the played value of second random

variable X_2.

3. Drawing of the value of normally distributed random variable.

Let us pause specially at one very frequently encountered problem: the drawing of the value of random variable X, distributed according to the normal law (it is shorter - "normal") with mathematical expectation m_x and root-mean-square deviation σ_x The density of distribution of random variable X takes the form:

$$f(x)=\frac{1}{\sigma_x\sqrt{2\pi}}e^{-\frac{(x-m_x)^2}{2\sigma_x^2}}. \tag{3.1}$$

According to general rule it was necessary to act thus: to construct the function of distribution F(x) and to find for it inverse function F^{-1}, then to this conversion to subject random number R from 0 to 1.

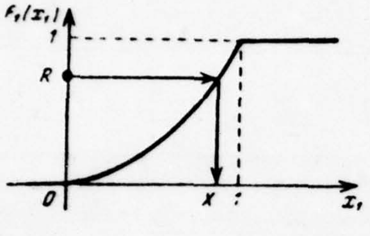

Fig. 8.11.

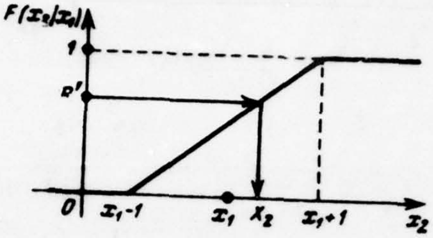

Fig. 8.12.

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However, conveniently to act otherwise: to pass from random variable X and another, to so-called "standardized/normalized" random variable:

$$Z = \frac{X - m_x}{\sigma_x}, \qquad (3.2)$$

to play the value of this random variable, and then to already from it find X. This is convenient because the mathematical expectation of value Z is equal to zero, and its root-mean-square deviation - to unity:

$$m_z = 0, \quad \sigma_z = 1,$$

it is necessary only one time and to forever find inverse function.

It is real/actual, let us designate the density of distribution

cf standardized value Z

$$f_{н}(z)=\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}. \qquad (3.3)$$

The normalized distribution function will be:

$$F_{н}(z)=\int_{-\infty}^{z}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz=0,5+\Phi(z), \qquad (3.4)$$

where

$$\Phi(z)=\frac{1}{\sqrt{2\pi}}\int_{0}^{z}e^{-\frac{z^2}{2}}dz$$

- the function of Laplace.

Plotted function $F_{н}(z)$ is given in Fig. 8.13. There by rifleman/pointers is shown obtaining random number X with density (3.3). Analytically this is written out in the form:

$$R=0,5+\Phi(Z),$$
$$Z=\Phi^{-1}(R-0,5), \qquad (3.5)$$

where Φ^{-1} - function, inverse Laplace functions Φ.

After playing the value of standardized/normalized random variable Z, let us pass from it to value X on the formula

$$X=\sigma_x Z+m_x. \qquad (3.6)$$

Thus, the value of normal random variable X with characteristics m_x, σ_x is played on the formula

$$X=\sigma_x\Phi^{-1}(R-0,5)+m_x, \qquad (3.7)$$

i.e. it is necessary to take random number R from 0 to 1, to subtract from it 0.5, to take from result inverse function of Laplace, to multiply on σ_x and to adjoin m_x.

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In the case when gaming of normal random variable is realized not by hand, but in machine, usually is applied another method, based on the central limit theorem of the probability theory. According to this theorem, during the addition of the sufficiently large number of the independent random quantities, congruent in their dispersions, is obtained random variable, distributed approximately according to normal law, moreover this law is that nearer to normal, the more random variables store/adds up. Experiment shows that for obtaining the virtually normal distribution of the sufficiently comparatively small number of terms. For example, during the addition a total of of six random numbers from 0 to 1 is obtained random variable, which with the accuracy, sufficient for the majority of applied problems, can be considered normal.

Hence appears this method of the drawing of normally distributed random variable X: to sum six random numbers from 0 to 1; to standardize this sum, i.e., to obtain standardized value Z, and then from it to pass to X on formula (3.6).

Let us convert the appropriate. Let us designate

$$V = R_1 + R_2 + R_3 + R_4 + R_5 + R_6,$$

where R_1, ..., R_6 - six independent copies of random number from 0 to 1. Let us find mathematical expectation, dispersion and the root-mean-square deviation of random variable V. According to the theorem of the addition of the mathematical expectations

$$M[V] = m_v = m_{r_1} + m_{r_2} + \ldots + m_{r_6},$$

where $m_{r_1}, \ldots, m_{r_6}$ — the mathematical expectations of values R_1, ..., R_6. It is obvious, they continually are identical and equal to 0.5, hence

$$m_v = 6\cdot 0{,}5 = 3.$$

The dispersion of random variable v let us find from the theorem of the addition of the dispersions:

$$D_v = D_{r_1} + D_{r_2} + \ldots + D_{r_6},$$

where $D_{r_1}, \ldots, D_{r_6}$ — the dispersion of values R_1, ..., R_6. It is known that the dispersion of random variable R, distributed with constant density on section (α, β), equal to

$$D_r = \frac{(\beta-\alpha)^2}{12};$$

in our case this will be $D_r = 1/12$, from which

$$D_v = 6 \cdot 1/12 = 1/2,$$

and the root-mean-square deviation

$$\sigma_v = \sqrt{D_v} = 1/\sqrt{2}.$$

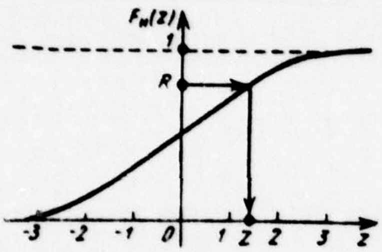

Fig. 8.13.

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We standardize value **V**, i.e., let us pass from it to value

$$Z=\frac{V-m_v}{\sigma_v}=(V-3)\sqrt{2}; \tag{3.8}$$

further from value Z, let us pass to the necessary to us value X on the formula

$$X=\sigma_x Z+m_x. \tag{3.9}$$

Substituting in this formula for Z its expression (3.8), and in (3.8), in turn, instead of **V** its expression

$$V=\sum_{i=1}^{6} R_i.$$

we will obtain finally:

$$X = \sigma_x \sqrt{2}\left(\sum_{i=1}^{6} R_i - 3\right) + m_x. \qquad (3.10)$$

In such a way as to play the value of normal random variable X with mathematical expectation m_x and root-mean-square deviation σ_x, it is necessary: to take six random number from 0 to 1, sum them up, from the sum subtract 3, result to multiply on $\sigma_x\sqrt{2}$ and to adjoin them m_x.

Now let us assume that we should play the value not of one, but several normally distributed random variables. If random variables are independent, problem simply is reduced to the realization of several tosses on the procedure described above. But if values are dependent, then with each following drawing it is must brother not simply the law of the distribution of next random variable, but its conditional law of distribution (when previous random variables they

took namely those values which were obtained drawing).

Example. System two random magnitude (X, Y) is distributed according to normal law with the characteristics:

$$m_x, m_y, \sigma_x, \sigma_y, r,$$

where r - a correlation coefficient. To construct the procedure of the drawing of the pair of values X, Y.

Solution. We develop first value of one of random variables, for example X, according to the procedure, described it is above for one normally distributed random variable with characteristics m_x and σ_x. The value of another random variable Y we develop already on conditional law of distribution [1] with mathematical expectation and the root-mean-square deviation:

$$\left.\begin{aligned} m_{y|x} &= m_y + r\frac{\sigma_y}{\sigma_x}(x - m_x), \\ \sigma_{y|x} &= \sigma_y\sqrt{1-r^2}, \end{aligned}\right\} \quad (3.11)$$

where x - the value, accepted by random variable X as a result of the previous toss.

FOOTNOTE 1. See, for example, [7]. ENDFOOTNOTE.

Let us note that from x in formulas (3.11) depends only mathematical expectation of conditional law, but not its root-mean-square deviation which with any x remains equal to $\sigma_y\sqrt{1-r^2}$

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4. Obtaining random number R from 0 to 1.

If the Monte-Carlo method is realized by hand (without the aid of machines), then for obtaining the random number from 0 to 1 most frequently are applied the so-called tables of random numbers. These tables are given in many management/manuals on mathematical statistics and computer technology (see for example, [18, 19]). Tables contain alternating in random order numerals 0, 1, 2, ..., 9.

During the composition of the tables, are accepted the measures for that, so that each of these numerals would be encountered approximately equally frequently and independent of others.

Using the table of random numbers, it is possible to easily play random number R from 0 to 1 with any number of decimal points after comma.

Let, for example, be required to obtain number R with four signs after comma. Let us turn to the table of random numbers and let us take from there any group of four series of the signs confronting, for example 7643. Let us consider that our random number took value as 0.7643. Following time when it is necessary to throw single toss, let us take following four numerals. Let they, for example, will be 3312 - that means that the following random number will be 0.3312, and so forth. It is possible to take the numerals, which stand not together, but through one; either in the beginning and at the end of the column, or rows - in a word, by any method, provided the principle of selection was not in any way connected with the values of numerals themselves.

For the drawing of random number R, it is possible to apply by hand not only tables of random numbers, but also other sensors, for example the disk, calibrated completely in the small divisions, labeled by numbers from 0 to 1 (Fig. 8.14). In the center of disk, is attached the well balanced arrow/pointer, given in the rotation/revolution, for example, by the electric motor, included by pushing of knob. After the release/tempering of rifleman/gunner's knob/button, it is stopped in random position, and its end indicates random number R.

If the Monte-Carlo method is realized not by hand, but by ETsVM, then for the selection of random number from 0 to 1 they can be applied both physical random-number transducers and the computational algorithms for obtaining the so-called "pseudorandom" numbers.

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Let us pause, first of all, at physical sensors. As a rule, they are based on the conversion of random signals (noises), either of natural or artificial origin. Let there be the random noise (i.e. randomly varying stress U(t) (Fig. 8.15), which we compare with certain fixed level u_0. This noise can be placed as the basis of the rule of the consumption/production/generation of random number from 0 to 1. Let us assume that ETsVM works in the binary code; then random number from 0 to 1 represents by itself the binary proper fraction, in which on each place are equally probable signs 0 or 1. Let us agree to consider that the next bit of random number will be 0, if for certain time interval T noise U(t) exceeded level u_0 even number once, and 1 - if odd. Now let us assume that n of such sensors work simultaneously and send random signs to 0 and 1 in n of the bits of the register of certain number R. Then, if time interval T to take is sufficient large then so that on it would be placed sufficiently many

fluctuations of noise U(t), then even and odd numbers of exceedance of level u_0 they will be encountered on the average equally frequently, and n- discharge binary number will be distributed approximately evenly on section 0.1.

It is possible to propose other principles of the formation of random numbers on the basis of one or the other physical random process; they all require equipment ETsVM by special random-number transducers. For that not specialized ETsVM, only incidentally drawn on the simulation of operations by the Monte-Carlo method, this equipment itself does not justify. Much more frequent during simulation by the Monte-Carlo method they use the so-called pseudorandom numbers. So are called the numbers, developed (computed) by machine itself according to certain rule (algorithm), constructed, so that signs 0 and 1 would be encountered on the average equally frequently, and, furthermore, so that the dependence both between the separate signs and between the formed of them multiple-digit numbers would be virtually absent. For obtaining the pseudorandom numbers, they use different methods. For example, it is possible to multiply two arbitrary n- marking binary numbers a_1 and a_2 and from product to take n of average/mean signs - this will number a_3; then multiply a_2 and a_3 and repeat procedure, etc.

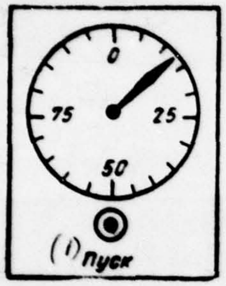

Fig. 8.14.

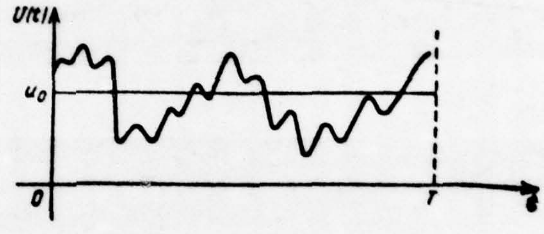

Fig. 8.15.

Fig. 8.14.

Key: (1). Launching/starting.

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There are methods of obtaining the pseudorandom numbers, based not on the multiplication of numbers, but on their shift/shear relative to each other on several discharges; after shift/shear is produced the additicn and then selection from sum n of average/mean signs and, etc. The different methcds cf obtaining the pseudorandom numbers are

described in detail in the special management/manuals (for example, see [15, 27]).

It must be noted that the pseudorandom numbers, strictly speaking random are not (entire/all their sequence can be predicted on the basis of initial material). In particular, any algorithm of the calculation of pseudorandom numbers is cyclic, i.e., through some large number Ts of the manufactured thus numbers, they unavoidably will begin to be repeated. However, if during the simulation of operation for us it is necessary to use a quantity of drawings less than Ts, this cyclic recurrence no value has.

At present during the simulation of operations by the Monte-Carlo method by ETsVM they usually use pseudorandom numbers, choosing one of the well inspected and checked algorithms, that ensures sufficient length of cycle, acceptable uniformity and the independence of numbers with comparative simplicity of their calculation. The advantages of pseudorandom numbers include the fact that they allow/assume the possibility of the secondary control error of the same realization of random process; other methods of the formation of random numbers (physical sensors) this possibility do not allow/assume.

5. Examples of the simulation of random processes by Monte-Carlo method.

In this paragraph will be examined some examples of the practical problems which, by virtue of their comparative complexity, are unattainable for analytical solution and require simulation by Monte-Carlo method. In each example we will construct the diagram of simulation, i.e., the sequence of calculations and single tosses, and also the method of treating the realizations.

Example 1. Technical equipment/device consists of three assemblies: u_1, u_2 and u_3. The exact work of assemblies u_1 and u_2 is unconditionally necessary for the work of equipment/device; assembly u_3 is intended for maintaining the normal mode of the work of assemblies u_1 and u_2. Equipment/device must work for a period of time τ. The time of the failure-free operation of each assembly is random; for assemblies u_1, u_2, u_3, it has density of distribution respectively $f_1(t)$, $f_2(t)$, $f_3(t)$. Available are two spare copies assemblies u_1 and three spare copies of assembly u_2. During the malfunction (failure) of assembly u_1 technical equipment/device is stopped at the random time, distributed with density $\varphi_1(t)$, after which assembly is substituted spare (if they still not all are spent), and the work of equipment/device is renewed. With the failure of assembly u_2, the equipment/device also is stopped at the random time, distributed with density $\varphi_2(t)$, assembly is substituted spare

(if such still are in the presence), after which the work of equipment/device it is renewed. If simultaneously do not work assemblies $\mathcal{U}_1$ and $\mathcal{U}_2$, the work of equipment/device is renewed only after is finished the replacement of last/latter assembly. If it left the system (it refused) assembly $\mathcal{U}_3$, it do not substitute, but the law of time allocation of the failure-free operation of assemblies $\mathcal{U}_1$ and $\mathcal{U}_2$ varies if to breakdown of assemtly $\mathcal{U}_3$ assembly $\mathcal{U}_1$: it studied time i_1, then the conditional density of distribution of the remaining time of the failure-free operation of assembly $\mathcal{U}_1$ will be $f_1(t/t_1)$; assembly $\mathcal{U}_2$ - $f_2(t/t_2)$.

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It is required to find the following performance characteristics of the equipment/device:

- probability of the exact work of equipment/device $p_n(t)$ as function of time;

- probability that the final failure of equipment/device of more earlily time τ will occur because of the deficiency of the spare assemblies $\mathcal{U}_1$;

- the mean time $\bar{t}_p$ of the operation of system, i.e., the mean

time which equipment/device will carry out in working state,

- the average number $\bar{y}_1$ of the spare assemblies U_1 which will be spent, and also the average number $\bar{y}_2$ and the spent spare assemblies U_2.

Solution. Since the laws of distribution, which figure in problem, are different from the exponential, to apply for describing the phenomenon the diagram of Markov processes we not can. We construct the diagram of simulation of random process by the Monte-Carlo method. *First of all, we determine by drawing* the time of the failure-free operation of assembly U_3 (which is not restored). For this, we find the function of the distribution

$$F_3(t) = \int_0^t f_3(t)\,dt.$$

we take random number R from 0 to 1 we subject to its conversion $F^{-1}_3(R)$ (Fig. 8.16) [1].

FOOTNOTE [1]. F^{-1}_3 - function, reverse F_3. ENDFOOTNOTE.

If as a result of this drawing value T_3 render/showed lesser τ, then we record/fix T_3 as torque/moment of the failure of assembly U_3; but if it turned out that $T_3 > \tau$, we consider that for time τ assembly U_3 did not fail. Let us assume that the first (more complex) variant took place, and assembly U_3 refused at torque/moment $T_3 < \tau$.

Let us consider four parallel axes 0t with one countdown (Fig. 8.17). On axis (1) we will note the state of the first assembly (by heavy line - "works", fine/thin - "refused"). On axis (2) also are noted the states of the second assembly, on axis (3) - the third, on axis (4) - a state of system as a whole ("works", "it does not work").

Since the torque/moment of failure T_3 of node/unit u_3 to us is known, then we can immediately fill axis (3). After this let us fill (1) and (2). Let us first play time T_1, during which will work the basic unit u_1 - for this we we will use the function of the distribution

$$F_1(t)=\int_0^t f_1(t)\,dt. \tag{5.1}$$

Further, let us play the time τ_1, during which this unit will be replaced spare. For this, we will use the function of the distribution

$$\Phi_1(t)=\int_0^t \varphi_1(t)\,dt. \tag{5.2}$$

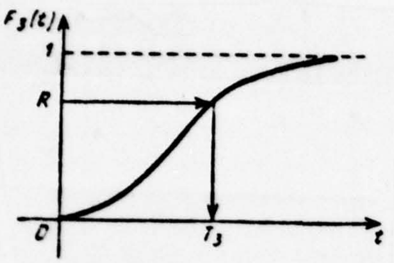

Fig. 8.16.

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If in torque/moment $T_1 + \tau_1$ the termination of this replacement the third unit still works ($T_1 + \tau_1 < T_3$), then we again develop the value of the operating time of the first spare unit T_1 with the help of function (5.1) and, after this - again the time of the replacement of this unit τ_1, with the help of function (5.2). Let us assume that the torque/moment of the termination of this replacement (as shown in Fig. 8.17) it render/showed after torque/moment T_3:

$$T_1+\tau_1+T_1'+\tau_1'>T_3. \qquad (5.3)$$

Then with the drawing of the new value of the time of the failure-free operation of unit Y_1 we must let us use no longer function (5.1) a the new conditional function of the distribution

$$F_1(t/0)=\int_0 f_1(t/0)\,dt, \qquad (5.4)$$

assuming that to the torque/moment of the failure of unit $\mathcal{U}_3$ the new unit $\mathcal{U}_1$ did not work (t_1 = 0). Let this played value be equal to T_1''.

Let us assume that, as shown in Fig. 8.17.

$$T_1+\tau_1+T_1'+\tau_1'+T_1''<\tau. \quad (5.5)$$

This means that at the torque/moment, noted in figure by asterisk, unit $\mathcal{U}_1$ left the system, to replace it already with something (a total of two spare units) and, which means, that at this moment finally refused whole equipment/device. Obviously further this point of news drawing not it is necessary.

We will be occupied axis (2), on which will be reflected the state of the second unit $\mathcal{U}_2$. For this axis let us lead the second, the analogous by the first, series of drawings, with that difference that the functions of the distributions by which converts random number R, will be others:

$$F_2(t)=\int_0^t f_2(t)\,dt, \quad (5.6)$$

$$\Phi_2(t)=\int_0^t \varphi_2(t)\,dt, \quad (5.7)$$

$$F_0(t/t_2)=\int_0^t f_2(t/t_2)\,dt. \quad (5.8)$$

Let us assume upon expiration of the time of the second replacement of τ'_2 we play value of T''_2 the operating time of the second spare unit Ψ_2, and it render/showed similar that torque/moment T_3 of breakdown of the third unit arrived for the operating cycle of the second unit.

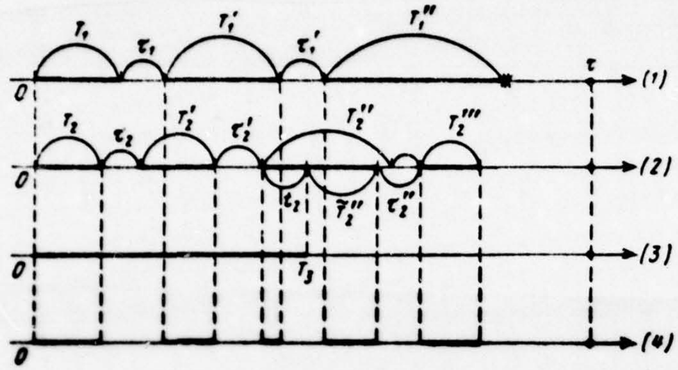

Fig. 8.17.

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Since during the malfunction of the third assembly of the condition of the work of the second unit they deteriorate, it is necessary into value T''_2 "to introduce correction" during the period of the failure-free operation of this copy u_2: to take into account that it already studied time t_2, and the residue/remainder of time T''_2 after torque/moment T_3 to play anew, already according to the changed (conditional) law of distribution (5.8).

The obtained thus value $\tilde{T}''_2$ it is necessary to add to already passed time t_2.

After this is developed the time of replacement τ''_2 this unit (according to the law (5.7)) and finally time $\tilde{T}'''_2$ of the failure-free work of the last/latter (the third) spare unit $\mathcal{U}_2$: since unit $\mathcal{U}_3$ already refused, then in this case we use law (5.8) with t_2 = 0. If the played value of time, in sum with all previously deposited/postponed on axis (2) times, it ends more to the right point, noted by asterisk, then, which means, that the reason for the failure of equipment/device in this case was the deficiency of the spare units $\mathcal{U}_1$. If the series of the intervals, deposited on axis (2), ends more left than the point, noted by asterisk - that means that in this realization as the reason for the failure of equipment/device was used the deficiency of the spare units $\mathcal{U}_2$.

Finally, let us fill last/latter axis (4), in which is reflected the work of equipment/device as a whole. According to condition the equipment/device works only in those torque/moments when work two units $\mathcal{U}_1$ and $\mathcal{U}_2$ simultaneously. Therefore on axis (4) we note by heavy line only those sections of time, for which the greasy/fatty sections of axes (1) and (2) coincide.

Thus, is played one realization of our random process. It goes without saying that if simulation is produced by ETsVM, any graphs,

axes and sections to construct not is necessary; drawing is ensured by bringing into action of the machine calculated algorithm which combines single tosses - drawings with the comparison between themselves of the torque/moments of realizing the different events (which occurred more earlily - the restoration/reduction of the first (the second) unit or breakdown of the third?).

Let us assume that in this or some other way you obtained a large quantity (N) of realizations of random process. Then, using the limit theorems of the probability theory and substituting the unknown probabilities by frequencies, but mathematical expectations by arithmetic means, we can approximately answer all the placed in problem questions.

Probability $p_{\text{ж}}(t)$ of the exact work of equipment/device at torque/moment t can be counted as follows: for each (the i-th) realization to introduce into examination the random function of time $X_i(t)$, which is equal to zero, when equipment/device does not work, and to unity - when works. The possible form of the separate realization of random function $X_i(t)$ is shown on Fig. 8.18. The probability of the exact work of equipment/device at torque/moment t is nothing else but the mathematical expectation of random function X(t) or, approximately, arithmetic mean of realizations $X_i(t)$:

$$p_n(t) \approx \frac{1}{N} \sum_{i=1}^{N} X_i(t). \qquad (5.9)$$

The possible form of probability $p_n(t)$ is shown on Fig. 8.19. The decrease of function $p_n(t)$ is connected with the fact that in the course of time increases the failure probability of the third unit and, furthermore, rise chances by the fact that spare units it will be insufficient.

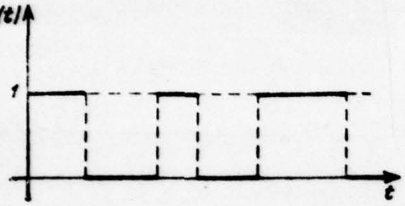

Fig. 8.18.

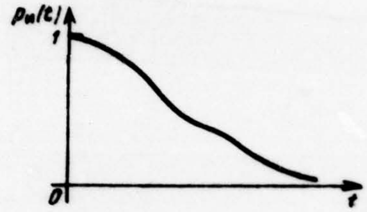

Fig. 8.19.

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Let us find probability that the failure of equipment/device will occur because of the deficiency of the spare units u_1. Let us consider event A, consisting in the fact that the failure of equipment/device of more earlily time τ will occur for this reason. Let us connect with each realization random variable Y_i, equal to unity if in this realization event A occurred, and zero - if did not occur. With the large number of realizations N, probability P(A) of event A is approximately equal to its frequency, but the latter there is nothing else but the ratio of the sum of all random variables to the number of realizations N:

$$P(A) \approx \frac{1}{N} \sum_{i=1}^{N} Y_i. \quad (5.10)$$

Let us determine the mean time t_p which equipment/device will carry out in working state. For this, it is necessary for each realization to determine its operating time $t_p^{(i)}$ – the sum of the lengths of all working sections of axis (4) to torque/moment τ - and to find their arithmetic mean:

$$\bar{t}_p \approx \frac{1}{N} \sum_{i=1}^{N} t_p^{(i)}. \qquad (5.11)$$

Finally, the average number $\bar{y}_1$ of spare units $\mathcal{U}_1$ which will be spent, will be located as arithmetic mean of the numbers of spent units $y_1^{(i)}$ for all realizations:

$$\bar{y}_1 \approx \frac{1}{N} \sum_{i=1}^{N} y_1^{(i)}, \qquad (5.12)$$

where $y_1^{(i)}$ – the number of spare units $\mathcal{U}_1$, spent in the i realization.

Analogously it is determined

$$\bar{y}_2 \approx \frac{1}{N} \sum_{i=1}^{N} y_2^{(i)}. \quad (5.13)$$

Thus, we constructed the pattern of the simulation of process by the Monte-Carlo method. Let us note one characteristic feature of method. In example 1, we assigned the mission of determining a total of of five values: $p_n(t)$, $P(A)$, $\bar{t}_p$, $\bar{y}_1$ and $\bar{y}_2$. However, the volume of calculations barely would be increased, if we wanted besides these five values to determine even a whole series of others, for example probability that both unit u_1 and u_2 will stand (not to work) is simultaneous, either the average/mean sense of the operating time of the first and second units, or the dispersion of the time of the exact work of equipment/device, or any other probabilistic characteristic of process. It is real/actual, during simulation by the Monte-Carlo method the lion fraction of time it occupies very simulation of realizations and only negligible portion/fraction - their treatment/working. Therefore, organizing the simulation of operation on EVM [ЭВМ - computer] by the Monte-Carlo method, always has sense to care about, "deducing" from the machine a little more of the information about each realization and to count the a little more characteristic parameters, without being limited to the calculation of the only one index of efficiency.

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Example 2. Let us consider the problem, similar to that that has already been mentioned in § 1. is produced shooting n by rockets at the area target of complex configuration (Fig. 8.20). The zone of fractures from one rocket represents by itself the circle of radius r. As a result of n of shots, will be struck some part S_n of target area (see the shaded range on Fig. 8.20) Ts, which composes some portion/fraction of full/total/complete target area:

$$U = \frac{S_n}{S_{\text{ц}}}.$$

In order to avoid the unnecessary overlaps of lethal areas, aiming n by rockets is produced not on one point, but on n to different points: O_1, O_2, ..., O_n. Are assigned the characteristics of missile dispersion the root-mean-square deviation along the axes Ox and Oy, equal to σ_x, σ_y. Systematic errors are absent, coordinates X, Y of each impact point are independent of each other and from coordinates of other impact points. It is required with the assigned location of aiming points O_1, O_2, ..., O_n to compute the following characteristics of the efficiency of the operation:

- average portion/fraction of the affected target area:

$$m_u = M[U];$$

- dispersion of the portion/fraction of the affected target area:

$$D_u = D[U];$$

- probability that will be struck not less assigned portion/fraction u of target area:

$$P(U > u);$$

- mathematical expectation $\overline{F}_0$ of the number of rockets which caused the target/purpose of no damage (they hit by).

Solution. If we make no simplifying assumptions about the form of target/purpose and lethal area, the analytical solution of stated problem is extremely complicated and it is it is virtually unrealizable; it will simpler solve by its Monte-Carlo method. Each realization will represent by itself the "bombardment" of target/purpose n by rockets, in which the impact points of rockets are played on toss. The simulation of each realization will consist of n of single tosses, plus the calculation of the affected area S_n

In each single toss is developed the impact point of one (the j-th) rocket, i.e., two random variables X_j, Y_j, distributed according to normal law with the characteristics

$$m_{x_j},\ m_{y_j},\ \sigma_x,\ \sigma_y,$$

where m_{x_j}, m_{y_j}— the coordinate of pcint O_j (correlation coefficient is equal to zero, since values X_j, Y_j according to condition are independent).

Let us assume that the simulation is produced by ETsVM. Then the most convenient method of the drawing of the pair of normal values X_j, Y_j will be described in § 3 additicn of several independent random numbers from 0 to 1 with the subsequent renormalization. With this method of the coordinate of the j impact pcint, they can be played on the formulas:

$$\left.\begin{aligned} X_j &= \sigma_x \sqrt{2}\left(\sum_{k=1}^{6} R_k - 3\right) + m_{x_j}, \\ Y_j &= \sigma_y \sqrt{2}\left(\sum_{k=7}^{12} R_k - 3\right) + m_{y_j}, \end{aligned}\right\} \quad (5.14)$$

where $R_1, R_2, \ldots, R_{12}$ - 12 separate independent copies of random number from 0 to 1 [1].

FOOTNOTE [1]. It goes without saying that for the drawing of the coordinates of each new impact point, it is necessary to take new of 12 random numbers. ENDFOOTNOTE.

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Let us assume that this stage of simulation is carried out, and we obtained n of impact points in this realization. Now it is necessary to count the affected area S_n for this (the i-th) realization. For this, it is necessary around each impact point to describe the circle of radius r and to count the area of that part of the target/purpose, which is covered at least with one of the circles. If drawing was produced by hand, it would be possible to determine this area by planimetry. During simulation in machine, they enter otherwise: entire/all target/purpose is divided into the large number of surface elements dS (Fig. 8.21) and for each of them is determined that is how its distance ρ_j from the impact point of the j rocket (j = 1, ..., n). If at least for one of the impact points this distance render/showed lesser than r (effective casualty radius), then area/site ds is considered affected, after which is produced the addition (integration) of affected area/sites ds_n on the entire target/purpose:

$$S_n = \sum_n ds_n.$$

Dale the obtained in the i realization value $S_{n}^{(i)}$ for the target area, we obtain the portion/fraction of the affected area in this realization:

$$U^{(i)} = \frac{S_{n}^{(i)}}{S_{n}}.$$

Incidentally with value $U^{(i)}$ for each realization, we compute $r_{0}^{(i)}$ — a quantity of rockets, a distance from the impact points of which to target/purpose it exceeds l (in this realization these rockets did not cause the damage to target/purpose). Having these data for the large number of realizations N, we can answer all the placed questions.

Average portion/fraction of the affected area:

$$m_{u} \approx \frac{1}{N} \sum_{i=1}^{N} U^{(i)}.$$

Dispersion of the portion/fraction of the affected area:

$$D_{u} \approx \frac{1}{N} \sum_{i=1}^{N} (U^{(i)})^{2} - m_{u}^{2}.$$

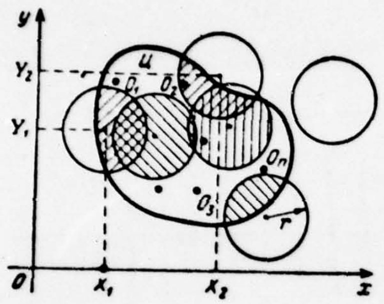

Fig. 8.20.

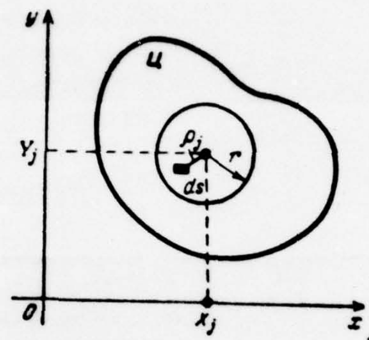

Fig. 8.21.

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Probability that the portion/fraction of the affected area will be not lesser than u, is determined as follows: with each realization is connected the number X_i, equal to one, if in this realization $U^{(i)} > u$, and zero, if $U^{(i)} < u$. Then

$$P(U > u) \approx \frac{1}{N} \sum_{i=1}^{N} X_i.$$

The mathematical expectation $\bar{r}_0$ of the number of rockets, which did not cause the damage to target/purpose, will be located from the formula

$$\bar{r}_0 \approx \frac{1}{N} \sum_{i=1}^{N} r_0^{(i)},$$

where $r_0^{(i)}$ – the number of rockets, which did not cause the damage to target/purpose in the i realization.

6. Determining the characteristics of stationary random process by Monte-Carlo method from one realization.

During the statistical simulation of operations, frequently it is necessary to meet the case, when the random process being simulated is stationary it occur/flow/lasts unlimitedly for long, having time-independent probabilistic characteristics.

As an example let us consider operation n- of the channel system of mass maintenance with m places in the turn the graph/count of states of which is shown on Fig. 8.22. Let us assume that the flow of claims, which translates system from state into state (from left to right), stationary, but not Poisson, for example the flow is palm with the arbitrary law of distribution f(t) of time interval T between claims. The servicing time of one claim is also distributed not according to exponential law, but according to arbitrary law

φ(t). Since the process, which takes place in system, non-Markov (flows of events - non-Poisson), then it cannot be described with the help of the standard mathematical apparatus of the Markovian processes - ordinary differential equations for the probabilities of states and algebraic equations - for the maximum probabilities of states. And generally, the attempt to describe this random process with the help of analytical dependences would lead to the excessively bulky of equipment, not justifying itself in practice. By the only virtually suitable method of the study of similar non-Markov systems is the simulation of process the Monte-Carlo method.

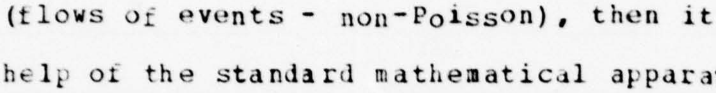

Fig. 8.22.

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If speech occurs about the study of the initial, unsteady period of functioning of system, then simulation is produced in the usual way - it is developed many realizations of process, and necessary to us probabilistic characteristics, for example, failure probability, the average number of occupied channels, the average number of claims

in turn, etc., are located by the treatment of "experimental" material as statistical average through many realizations.

However, when speech occurs about the study not of transient, initial period, but steady, steady-state state, attained with $t \to \infty$, the situation changes. In fact, when simulating steady-state random processes we can usually use not a great number of realizations, but one sufficiently long realization.

Here the probability characteristics of the random process of interest to us can be obtained not as the means for a large number of realizations, but as the time means for one sufficiently long realization.

Strictly speaking, stability of the process alone is insufficient for this. The process should also posess the so-called ergodic property. In an elementary interpretation the essence of this property is that the limiting mode which is established in the system often some time of its operation does not depend on what the initial conditions and the initial period of the system's operation were - each separate realization is a sort of "plenipotentiary representative" of the entire class of realizations. This means that whatever realization we may choose, with $t \to \infty$ we will obtain process with one and the same by characteristics.

It is possible to give an example of the process of stationary, but not possessing ergodic property. Let, for example, be examined the system with the graph/count of states, shown on Fig. 8.23. All flows of the events, which translate system from state into state, we consider stationary. Let at the initial moment t = 0 system are be located in state S_0; from it it can pass either into state S_1 or in S_3. After passing into state S_1, system will begin "to circulate" due to states S_1 and S_2. Because of the stability of the flows of the events, calling this circulation, in sufficient time of probability $p_1(t)$ and $p_2(t)$ states S_1 and S_2 they will become constants, but the process of circulation - stationary:

$$p_1(t) \to p_1 = \text{const}; \quad p_2(t) \to p_2 = \text{const}.$$

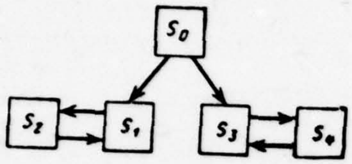

Fig. 8.23.

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But if from state S_0 system passed not in S_1, but in S_3, then it will circulate not due to states S_1, S_2, but due to states S_3, S_4: the probabilities of these states will also approach constants:

$$p_3(t) \to p_3 = \text{const}; \quad p_4(t) \to p_4 = \text{const},$$

but already for others, than p_1 and p_2.

Thus, in the given example the process, which takes place in system, will be stationary, but ergodic, and its probabilistic characteristics depend substantially on initial period (initial behavior of system). It is clear that the simulation of this process with the help of one (at least and by very long) realization insufficient for obtaining its probabilistic characteristics.

Fortunately, ergodic random processes to happiness, the ergodic random processes in practice are encountered more frequently than nonergodic ones and, as a rule, the simulation of one realization makes it possible to obtain all the probabilistic characteristics. In particular, ergodic prove to be the processes, which take place in the systems the graph/count of states of which is related to the pattern of "death and multiplication", as shown, for example, on Fig. 8.22. Here system can through some number of step/pitches pass from each state of each another and the "splitting/fission" of the process, similar occurring in system with graph/count Fig. 8.23, does not occur.

If system has an infinite multitude of possible states, then, we know that even with the stability of all flows of events, maximum conditions/mode with $t \to \infty$ it can not exist - we this saw based on the example of the system of mass maintenance with the unlimited turn (see § 6 of Chapter 5), where with $t \to \infty$ turn when $\varkappa > 1$ increases unlimitedly. However, if maximum conditions/mode exists, then during the simulation of process by the Monte-Carlo method it is possible to be bounded to one realization.

To demonstrate the existence of maximum conditions/mode we,

strictly speaking, can only for a Markov system, and simulation by the Monte-Carlo method is applied, as a rule, to systems non-Markov. However, with the help of indirect reasonings frequently and of this case it is possible to be convinced of the existence of maximum conditions/mode.

For the explanation that presented, let us consider the example, which relates to simulation by the Monte-Carlo method the operation of non-Markov system of mass maintenance with turn.

Example. There is two-channel ($n = 2$) SMO with turn. The number of places in turn $m = 3$; the claim, which came at the torque/moment when all the three places in turn are occupied, obtains failure and leaves system. The flow of claims - Palmov, i.e., time intervals between claims represent by themselves the independent random quantities, distributed according to one and the same (nonindicative) law $f(t)$ (Fig. 8.24). Servicing time of one claim - also random variable, distributed according to nonindicative law $\phi(t)$ (Fig. 8.25), different from $f(t)$, but identical for all claims.

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It is required, simulating the work of SMC with the Monte-Carlo method and disposing of only one by long realization, to consider the

approximately maximum characteristics of system (with $t \to \infty$):

- probability of states (probability of the fact that they will be occupied 0.1, 2 channels; the probability of the fact that in turn they will be located 0, 1, 2, 3 claims);

- the average number of occupied channels;

- mean latency of claim in turn; the dispersion of latency of claim in turn;

- failure probability (that which claim will leave SMO not serviced).

To construct the pattern of simulation and the set-up of processing its results.

Solution. The graph/count of the states of system takes the form, shown on Fig. 8.26. The number of states is certain; from each state it is possible to pass into each; the flows of events, which translate system from state into state, are stationary (although nonPoisson); from this we consist that the system possesses ergodic property and simulation on one realization is possible.

Let us assume for simplicity that at the initial moment (t = 0) the system is in state S_0 (is free) [1].

FOOTNOTE [1]. This does not have a value, since maximum conditions/mode does not depend on initial state.

ENDFOOTNOTE.

Let us begin simulation from the fact that let us play on axis 0t the flow of claims, i.e., a series of the random points t_1, t_2, t_3, ... - the torque/moments of the arrival of the corresponding claims - the first, the second and the like (Fig. 8.27).

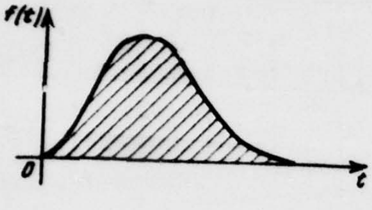

Fig. 8.24.

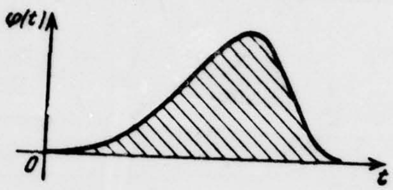

Fig. 8.25.

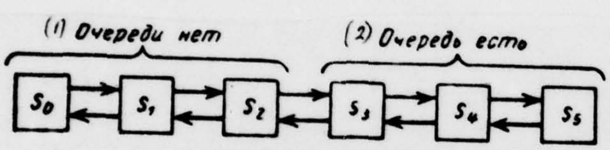

Fig. 8.26.

Key: (1). There is no turn. (2). Turn is.

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The drawing of the flow of claims is produced as follows. Is constructed the function of random number distribution T - interval between the claims:

$$F(t)=\int_0^t f(t)\,dt \tag{6.1}$$

and it is developed the value of random variable T_1, as this described in § 2; for this is taken the function, reverse F, from random number R from 0 to 1: $T_1 = F^{-1}(R)$.

Distance T_1 is plot/deposited from the origin of coordinates; is obtained torque/moment t_1 of the arrival of the first claim. Then the procedure of drawing is repeated (it goes without saying that already with other R) and the new value T_2 is plot/deposited from τ_1, is obtained torque/moment t_2 of the arrival of the second claim, and so forth [1].

FOOTNOTE [1]. During simulation in machine more convenient not to construct the chain/network of the arrivals of claims previously, but "to supply" them to SMO on one, with arrival; for the purpose of convenience in the explanation we assume that the claims are developed previously. ENDFOOTNOTE.

Thus we will construct the chain/network of the torque/moments of the arrival of claims (Fig. 8.27). It goes without saying that this

chain/network must be done sufficient long, after playing, in any case, not the less several hundred values of random variable T.

It is represented the procedure of simulation with the help of pictorial diagram (Fig. 8.28). Above we will place time axis (0) with the noted on it torque/moments of the admission of claims. Below it we will place an additional five axes: (1), (2), (3), (4), (5). On axes (1) and (2) we will represent the states of first and second channels (greasy/fatty feature - "it is occupied", fine/thin - "is free"). On axes (3), (4), (5) we will represent the 5 tests of the first, second, and third doses in turn (heavy line - "occupied", - fine line - "free"). Everything of five axes have the same count-down, as axis (0).

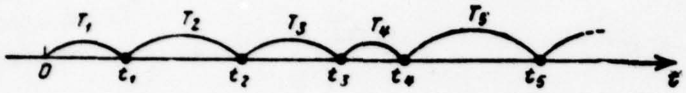

Fig. 8.27.

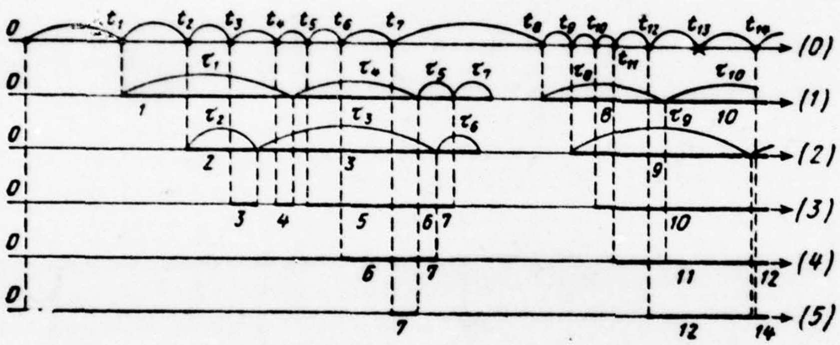

Fig. 8.28.

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To torque/moment t_1 - arrival of the first claim - all channels and all the places in turn are free. At torque/moment t_1, comes the first claim and it occupies the first channel. How long it will be occupied - it is solved by drawing. For this, we will subject random number R (it goes without saying that new) to conversion $\Phi^{-1}(R)$, where Φ the function of time allocation of the maintenance:

$$\Phi(t)=\int_0 \varphi(t)\,dt. \quad (6.2)$$

The first played value of servicing time τ we designate τ_1 and plot on axis (1) from point with abscissa t_1, noting it by heavy line (Fig. 8.28). At the moment of arrival t_2 of the second claim, the first channel is still occupied; claim occupies second channel. Let us play still one value τ, let us designate it τ_2 and will plot by heavy line on axis (2) from point with abscissa t_2.

Claim t_3, which came at the torque/moment when both channel are occupied, stops in turn, it occupies in it the first place (axis (3)) it awaits to that torque/moment when is freed one of the channels. In our case more earlily is free/released channel (2) - at this moment the point from axis (3) jumps to axis (2) - and again is developed the servicing time τ_3 of this claim. On axis (2) is constructed new greasy/fatty section, and axis (3) it is continued for fine/thin line - the place in turn it is free.

We will not continue the detailed description of the procedure of the drawing of realization - it is sufficiently clear from Fig. 8.28. In this figure against each section of the employment of

channel (place in turn) for convenience in the treatment is written the number of the claim, which occupies this place; it is possible to trace, as claim "travels" from the last/latter places in turn to the first, then - for maintenance. The claim, which obtained failure, is noted by the asterisk (it leaves SMO not serviced).

Let us assume that the simulation of realization have is continued we sufficiently for long (so for long which the effect of initial conditions already ceases to manifest itself). Let us look as by this realization to determine those interesting us the probabilistic performance characteristics of SMC. Probabilities p_0, p_1, $\tilde{p}_2$ that they will be occupied with 0, 1, 2 channels [1], let us find as follows.

FOOTNOTE [1]. Designation $\tilde{p}_2$ is introduced because this probability does not coincide with earlier (see Chapter 5) the introduced probability p_2, but is equal to $\tilde{p}_2 = p_2 + p_3 + p_4 + p_5$. ENDFOOTNOTE.

Let us divide entire axis 0t into sections with respect to the number of occupied channels. The sections of time, on which is occupied not one channel, let us note by numeral 0, one channel - by numeral 1, two channels - by numeral 2. On the large section of time T, let us add the lengths of all sections, by marked zero - we will obtain T_0; the sum of the lengths of all sections, noted by one, will be T_1,

pair - T_2.

It is obvious,

$$T_0 + T_1 + T_2 = T.$$

With large T of probability p_0, p_1 and $\tilde{p}_2$, they will be approximately equal to the ratios of the corresponding times to the total time:

$$p_0 \approx T_0/T; \quad p_1 \approx T_1/T; \quad \tilde{p}_2 = T_2/T. \qquad (6.3)$$

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Let us note that section T it is expedient to count off not from the very beginning of the process where still manifests itself the effect of initial conditions, but from more distant for time torque/moment 0', where the effect of initial conditions already in practice ceases to manifest itself.

Let us find probabilities $\hat{p}_0$, $\hat{p}_1$, $\hat{p}_2$, $\hat{p}_3$ that in turn they will stand 0, 1, 2, 3 claims. Let us again decompose the large section of time axis T into the parts, marked $\hat{0}$, $\hat{1}$, $\hat{2}$, $\hat{3}$, on which in turn it is located with respect to 0, 1, 2, 3 claims. Store/adding up the lengths of all equally marked sections and Dale sums T_i on T, we will obtain:

$$p_0 \approx T_0/T;\quad p_1 \approx \hat{T}_1/T;\quad \hat{p}_2 \approx \hat{T}_2/T;\quad \hat{p}_3 \approx \hat{T}_3/T. \qquad (6.4)$$

The average number of occupied channels $\bar{z}$ will be obtained in the usual way as mathematical expectation of discrete random variable Z - number of occupied channels:

$$\bar{z} = 1 \cdot p_1 + 2 \cdot \tilde{p}_2 = p_1 + 2\tilde{p}_2. \qquad (6.5)$$

Mean latency of claim in turn $\bar{t}_{ож}$ we find as follows: let us consider a series of the claims, which acted on the large section of time T the torque/moments

$$t_k,\ t_{k+1},\ \ldots,\ t_{k+i},\ \ldots,\ t_{k+N},$$

and for each of them let us directly count latency in turn $t_{ож}^{k+i}$, equal to zero, if (k + i) claims it was immediately accepted for the maintenance (or was obtained failure), and the sum of latople this claim for different axes ((3), (4) and (5)), if it stood in turn. Mean latency of claim in turn approximately will be located as arithmetic mean of these times:

$$\bar{t}_{ож} \approx \frac{1}{N} \sum_{i=0}^{N} t_{ож}^{(k+i)}. \qquad (6.6)$$

If us interests not the simply mean time of waiting, and

conditional mean time, calculated when the claim was accepted for maintenance, then arithmetic mean latople is computed not for all claims, but only for those that were serviced.

The dispersion of latency will be located with analogous form as arithmetic mean of the squares of latople minus the square of the mean time of waiting:

$$D[T_{\text{ож}}] \approx \frac{1}{N}\sum_{i=0}^{N}\left(t_{\text{ож}}^{(k+i)}\right)^2 - \bar{t}_{\text{ож}}^2. \qquad (6.7)$$

Finally, failure probability will be located on large section of time T as ratio of number N* of the claims, marked with asterisk (obtained failure), to the total number N of the claims, which acted for this time:

$$P_{\text{отк}} \approx \frac{N^*}{N}. \qquad (6.8)$$

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7. Estimation of accuracy of characteristics obtained by Monte-Carlo method. Necessary number of realizations.

The Monte-Carlo method is based on the limit theorems of the probability theory, which claim that with the large number of experiments N the frequency approaches its probability, and arithmetic mean of the observed values of random variable - to its mathematical expectation. Using the Monte-Carlo method, we, after producing the large number of experiments (realizations), approximately replace the probability of event with its frequency, and mathematical expectation - by arithmetic mean.

Logically does get up a question - how great will be the error, which appears from this approximate replacement? And how must be the number of realizations N, so that this error with practical authenticity would not leave beyond given limits? In other words, arises the question concerning the evaluation of the accuracy of the characteristics of random phenomenon, obtained by the Monte-Carlo method.

With answer/response to these questions, we will be based on the central limit theorem of the probability theory. According to this theorem, with the large number of experiments N their average result (frequency P^* of event A or arithmetic mean $\overline{X}$ of the observed values of random variable X) is distributed approximately according to

normal law. Let us give the relating here formulas.

1. Law of allocation of frequencies of event with large number of experiments.

If is produced the large number N of the independent experiments, in each of which event A appears with probability p, then frequency A

$$P^* = \frac{M_A}{N} \quad (7.1)$$

(where M_A the number of appearances of event A in N experiments) is distributed approximately according to normal law, with the mathematical expectation

$$m_{p^*} = p \quad (7.2)$$

and the root-mean-square deviation

$$\sigma_{p^*} = \sqrt{\frac{p(1-p)}{N}}. \quad (7.3)$$

2. Law of the distribution of arithmetic mean with the large number of experiments.

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If is produced the large number N of the independent experiments in which random variable X takes the values:

$$X_1,\ X_2,\ \ldots,\ X_N, \tag{7.4}$$

the arithmetic mean of these values:

$$\overline{X}=\frac{1}{N}\sum_{i=1}^{N}X_i \tag{7.5}$$

it is distributed approximately according to normal law, with the mathematical expectation

$$m_{\overline{x}}=m_x \tag{7.6}$$

and the root-mean-square deviation

$$\sigma_{\overline{x}}=\frac{\sigma_x}{\sqrt{N}}, \tag{7.7}$$

where m_x, σ_x mathematical expectation and the root-mean-square deviation of random variable X.

Being based on these laws of distribution and formulas, we can place and solve several problems, which relate to the accuracy of the Monte-Carlo method.

Problem 1. Is produced N of independent experiments (realizations), in each of which event A appears with probability p. As a result of these experiments, is obtained frequency P* of event A. To find probability that frequency P* differs from probability p no more than to assigned magnitude $\varepsilon > 0$.

Solution. Counting number N sufficient large, in order to set/assume frequency P* that distributed according to normal law with characteristics (7.2), (7.3), we will obtain:

$$P(|P^* - p| < \varepsilon) = 2\Phi\left(\frac{\varepsilon\sqrt{N}}{\sqrt{p(1-p)}}\right), \tag{7.8}$$

where Φ the function of Laplace [1].

FOOTNOTE [1]. The values of the function of Laplace see in Table 1 of application/appendix. ENDFOOTNOTE.

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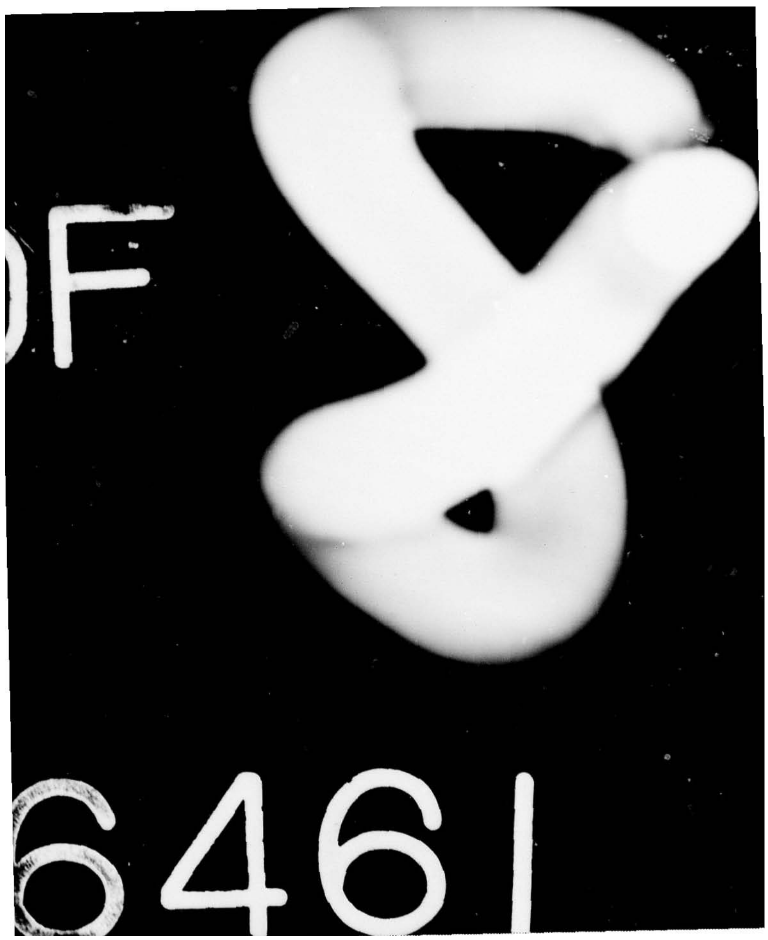
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Example 1. Produced N = 1000 independent experiments, in each of which event A appeared with probability p = 0.3. To find probability that the obtained with this frequency P* of event A differs from probability less than on $\varepsilon = 0{,}02$.

Solution. On formula (7.8) we have:

$$P(|P^* - 0{,}3| < 0{,}02) = 2\Phi\left(\frac{0{,}02 \cdot 31{,}6}{0{,}459}\right) = 2\Phi(1{,}38) \approx 0{,}83.$$

Thus, if probability p of event A to us is known, we can consider the accuracy of the determination of this probability from frequency P* and the dependence of this accuracy on the number of experiments N. Misfortune in the fact that probability p to us is unknown: indeed and themselves experiments we ran in order it to find. However, for estimating the accuracy of the Monte-Carlo method to us not very substantial to know a precise value of probability itself p - into the right side of formula (7.8) by it is possible to substitute tentative value, after taking instead of p, for example, frequency P* of event A in this set of experiments.

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Thus, we solved direct problem of the evaluation of the accuracy of the determination of probabilities by the Monte-Carlo method: if is known the number of experiments N and the tentative value of probability p, we can find probability that frequency P* will deviate from probability not larger than by the assigned magnitude ε.

Let us place now the inverse problem: how many experiments N it is necessary to produce in order with practical confidence to expect that the frequency will deviate from probability no more than by the assigned magnitude?

Problem 2. There is conducted a series of the independent experiments, in each of which event A appears with probability p. How must be the number of experiments (realizations), so that with the assigned, sufficiently high probability Q it would be possible to expect that frequency P* of event A will deviate from its probability p less than on ε?

Solution. Let us assign any sufficient to close to unity by the value of probability Q - let us name it the "level of confidence". If probability that the frequency and probability diverge less than by ε, will be Q or more, let us consider problem solved. In practice the level of confidence Q is chosen by any circular value, close to one, for example, 0.95 either 0.99 or 0.995 and so forth, depending on the importance of the problem which we pursue. Let us assume that probability Q it is assigned. Let us equate to this value of Q the right side of equality (7.8):

$$2\Phi\left(\frac{\varepsilon\sqrt{N}}{\sqrt{p(1-p)}}\right)=Q \qquad (7.9)$$

is solved equation (7.9) relative to N:

$$\Phi\left(\frac{\varepsilon\sqrt{N}}{\sqrt{p(1-p)}}\right)=\frac{1}{2}Q, \quad \frac{\varepsilon\sqrt{N}}{\sqrt{p(1-p)}}=\Phi^{-1}\left(\frac{1}{2}Q\right), \qquad (7.10)$$

where Φ^{-1} - function, inverse functions of Laplace. We hence obtain formula for the number of experiments N:

$$N=\frac{p(1-p)}{\varepsilon^2}\left[\Phi^{-1}\left(\frac{1}{2}Q\right)\right]^2. \qquad (7.11)$$

If on formula (7.11) N proves to be whole, it must be rounded off to large side to the nearest whole.

For the calculations on of formulas (7.11) is convenient to have available the table of the values of function $[\Phi^{-1}(\frac{1}{2}Q)]^2$. Table 7.1 gives corrected values of this function for some, most typical

values of the level of confidence Q.

Tables 7.1.

Q	0,80	0,85	0,90	0,95	0,96	0,97	0,98	0,99	0,995	0,999	0,9995	0,9999
$\left[\Phi^{-1}\left(\frac{1}{2}Q\right)\right]^2$	1,64	2,08	2,71	3,84	4,21	4,49	5,43	6,61	7,90	10,9	12,25	15,2

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Example 2. Is produced a series of independent experiments (realizations), in each of which is recorded the appearance or the nonappearance of event A whose probability p=0.2 how many experiments must be produced so that the frequency P* of event A with probability (level of confidence) Q=0.95 would differ from p not larger than on $\varepsilon = 0{,}01$?

Solution. Number of experiments N we calculate by formula (7.11). On table 7.1 for Q=0.95 we find

$$\left[\Phi^{-1}\left(\frac{1}{2}Q\right)\right]^2 = 3{,}84.$$

Substituting in formula (7.11) we will obtain:

$$N = \frac{0{,}2\cdot 0{,}8}{0{,}01^2}\cdot 3{,}84 \approx 6140.$$

i.e. for the reliable ($Q=0.95$) determination of probability $p=0.2$ KPO for frequency with error not more than 0.01 (i.e. within limits of 0.19-0.21) is required to carry out more than 6000 realizations [1].)

FOOTNOTE [1]. The value of probability p, entering formula (7.11), in practice it is possible to take tentatively, in frequency in the first series of realizations, refining it with the accumulation of material. ENDFOOTNOTE.

Problem 3. Is produced N of the independent experiments, in each of which is observed the value of random variable X, which has mathematical expectation m_x and root-mean-square deviation Is computed arithmetic mean of the observed values of random variable X:

$$\overline{X}=\frac{1}{N}\sum_{i=1}^{N}X_i. \tag{7.12}$$

To find probability that arithmetic mean $\overline{X}$ will deviate from mathematical expectation m_x less than by the assigned magnitude ε:

$$P(|\overline{X}-m_x|<\varepsilon).$$

Solution. On the basis of central limit theorem, considering the number of experiments large, it is possible to claim that random variable $\overline{X}$ is distributed normally, with characteristics (7.6) and

(7.7). Hence

$$P(|\bar{X}-m_x|<\varepsilon)=2\Phi\left(\frac{\varepsilon}{\sigma_{\bar{x}}}\right)$$

or

$$P(|\bar{X}-m_x|<\varepsilon)=2\Phi\left(\frac{\varepsilon\sqrt{N}}{\sigma_x}\right). \qquad (7.13)$$

On formula (7.13) can be estimated the accuracy of the determination of mathematical expectation from arithmetic mean.

Example 3. Is produced N=1600 the independent experiments, in which are observed the values of random variable X with characteristics $m_x=2$ and $\sigma_x=1$

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To find the probability of the fact that arithmetic mean of the observed values of random variable X will differ from its mathematical expectation less than to 0.05, i.e., it will be included in interval by 1.95-2.05.

Solution. Through formula (7.13), using table 1 of application/appendix, we find:

$$P(|X-m_x|<0{,}05)=2\Phi\left(\frac{0{,}05\cdot 40}{1}\right)=2\Phi(2)\approx 0{,}954.$$

Let us note that for estimating the accuracy of the determination of mathematical expectation m_x by the Monte-Carlo method is not required previously to know the quite mathematical expectation of random variable, then substantial to know its root-mean-square deviation σ_x, which enters in the right side of formula (7.13).

Usually in practice, beginning the simulation of random phenomenon by the Monte-Carlo method, we know either mathematical expectation or root-mean-square deviation of which interests us random variable. However, for the approximate estimate of the accuracy of simulation, it is possible in the first approximation, instead of σ_x to use its statistical evaluation, obtained in series itself from N of the realizations:

$$\sigma_x \approx \sqrt{\frac{1}{N}\sum_{i=1}^{N} X_i^2 - \bar{X}^2}, \qquad (7.14)$$

where $\bar{X}$ - arithmetic mean. If accuracy render/shows insufficient, one should continue testings, introducing into average the quadratic appropriate corrections with an increase in the number of realizations.

Problem 4. Is produced a series of the independent experiments above random variable X. How much it is necessary to do experiments in order with the assigned probability (level of confidence) Q to

expect that arithmetic mean $\overline{X}$ of the observed values of random variable will deviate from its mathematical expectation not more than on ε?

Solution. Let us place the right side of formula (7.13) equal to the level of the confidence Q:

$$2\Phi\left(\frac{\varepsilon\sqrt{N}}{\sigma_x}\right)=Q \qquad (7.15)$$

is solved equation (7.15) relative to N. We will obtain:

$$N=\left(\frac{\sigma_x}{\varepsilon}\right)^2\left[\Phi^{-1}\left(\frac{1}{2}Q\right)\right]^2, \qquad (7.16)$$

where $\left[\Phi^{-1}\left(\frac{1}{2}Q\right)\right]^2$ — - function, given in Table 7.1.

Example of 4. Are run the experiments above random variable X for purpose of approximately determining its mathematical expectation m_x. The root-mean-square deviation of random variable X, evaluated preliminarily (on the first series of experiments) with respect to formula (7.14), is approximately equal to $\sigma_x \approx 0.1$. Which number of experiments N is necessary so that (with the level of confidence Q=0.99) arithmetic mean $\overline{X}$ of the observed values of random variable X would differ from its mathematical expectation not more than on $\varepsilon = 0.01$?

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Solution. Using Table 7.1, for Q=0.99 we find:

$$\left[\Phi^{-1}\left(\frac{1}{2}Q\right)\right]^2=6,61;\ \left(\frac{\sigma_x}{\varepsilon}\right)^2=100.$$

Hence on formula (7.16)

$$N=100\cdot 6,61=661.$$

In conclusion let us pause briefly at the estimation of the accuracy of determining the characteristics of stationary random function from one realization (see §6). Since here there is no many realizations, and there is only one long realization, do arise the natural questions:

- which error in determination of the characteristics of random process from one realization of length T?

- which must be the length of realization T, so that with this level of confidence Q error would not exceed datum ε?

The exact solution of these problems does not simply require fine/thin reasonings. rough-approximately to these questions it is possible to answer, after reducing them to the questions, already solved for many realizations, if we conditionally equate on accuracy

one long realization of duration T to many realizations of length T' of the same common/general/total duration:

$$T=NT',$$

where the length of realization T' is defined as such time for which the correlation between the values of the random function X(t) being investigated becomes negligible.

In practice during the simulation of random process on one realization, frequently does appear the question: the pore whether already to stop? Did become already stable the probabilistic characteristics of process? In such cases instead of the tedious estimation of the accuracy of simulation it is possible to use following rough method: to sharply change the initial conditions by which is produced the simulation (for example, to assume that at the initial moment not "all channels are free", but "all channels are occupied") and to repeat simulation by the changed initial conditions. If in this case on the sufficiently distant from beginning sections of time are obtained virtually the same probabilistic characteristics of process, this good evidence in favor of the fact that to them it is possible to entrust.

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Are worked out the special mathematical methods, intended for the proof of solutions under conditions of indeterminancy/uncertainty. In some, simplest cases these methods make it possible actually to find and to select optimum solution. In the more complex cases these methods supply/deliver the auxiliary material, which makes it possible deeper to be dismantle/selected at complex situation and to consider each of the possible solutions from different (sometimes contradictory) points, to weigh its advantages and deficiency/lacks and in the final analysis to make a decision, if not singularly correct, then, at least, to end thought out.

It is necessary to consider that when selecting of solution under conditions of indeterminancy/uncertainty is always unavoidable the cell/element of arbitrariness and, which means, that risk. The insufficiency of information is always dangerous, and for it it is necessary to pay. However, under conditions of complex situation, it is always useful to present the versions of solution and their possible consequences in such form, in order to to do arbitrariness of selection by less rough, and risk - minimum.

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In a number of the cases, the problem of making of decision under conditions of indeterminancy/uncertainty is placed in this

form: by which value it is possible to pay for the missing information so that the economic effect of an entire operation would be maximum?.

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With the problems of making of decisions under conditions of indeterminancy/uncertainty occupies theory of games and statistical solutions.

In this chapter is stated some basic information from this region. For more detailed familiarization can be recommended works [24, 25, 29].

2. Object/subject of the theory of games. / basic concepts.

During the solution of a series of the practical problems of operations research (in the region of economics, of military science, etc.) it is necessary to analyze the situations, in which they collide two (or are more) quarreling sides, which pursue different target/purposes, the result of any measure of each of the sides depending on which modus operandi it will select enemy. Such situations we will call conflicting situations.

Examples of conflicting situations are very varied. Any situation, which store/adds up in the course of military actions, belongs to conflicting: each solution in this region must be accepted taking into account the conscious counteraction of reasonable enemy. To the same category belong the situations, which appear when selecting of the weapon system, methods of its combat employment and generally during planning of combat operations. A series of situations in the region of economics (especially in the presence of capitalist competition) also belongs to conflicting; in the role of the fighting sides, come forward commercial firms, industrial enterprises, trusts, monopolies, etc. Are encountered conflicting situations also in legal procedure, sport and in other fields of human activity.

Need to analyze such situations caused to life special mathematical apparatus - theory of games. The theory of games is mathematical theory of conflicting situations. Problem of this theory - consumption/production/generation of recommendations regarding the rational modus operandi of the participants of conflict.

Each directly undertaken from practice conflicting situation is very complex, and its analysis is hinder/hampered by the presence of many attendant, unessential factors. In order to do possible mathematical analysis of situation, it is necessary to be distracted

from these secondary factors and to construct the simplified, schematized model of situation. This model we will call game.

From real conflicting situation the game differs in terms of the fact that it is conducted according to the completely specific rules. Humanity since olden times uses such formalized models of conflicts - "games" literally (cartridge, chess, card games, etc.). All these games bear the character of the competition, which occurs according to known rules and which is finished "with one or the other player's conquest" (by gain).

Such formalized games represent by themselves the most convenient material for illustration and mastering basic concepts of the theory of games.

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This was reflected also in its terminology: both sides, participating in conflict, are conditionally named by "players", the issue of conflict - by "gain", etc.

In game can collide interests of two or more enemies; in the first case the game is called, "paired", in the second - "multiple". The participants of multiple game can form coalitions (constants or

time/temporary). Multiple game with two constant coalitions is converted into paired. Great practical value have paired games; we will be bounded to the examination only of such games.

Let there be the paired game И, in which they participate two players A and B with opposite interests [1].

FOOTNOTE [1]. This formal condition, obviously, gives no real advantages to player A. ENDFOOTNOTE.

Hearth "game" let us understand the measure, which consists of a series of actions or "courses" of sides A and B. So that the game could be subjected to mathematical analysis, must be clearly formulated the rules of game, i.e., the system of conditions, regulating:

- possible versions of the actions of the players,

- a volume of the information of each side about behavior another,

- a result (issue) of game, to which leads each this set of courses.

This result (gain or loss) not at all does not always have quantitative expression, but it is usually possible, at least conditionally, to express by its number (for example, in checkered game gain to ascribe value of 1, to loss - 0, draw - 1/2).

Game is called zero-sum game, if one player wins exactly as much, as loses another, i.e., the sum of the gains of sides is equal to zero. In zero-sum game, the interests of enemies are directly opposite. Here we will examine only such games.

Let us designate a player's gain A, a b - player's gain B in zero-sum game. Since a=-b, then during the analysis of this game there is no need for examining both these number-sufficient to examine gain of one of the players; let this will be, let us say, that A. Subsequently we, for convenience in the presentation, side A will conditionally name "we", and side B - "enemy" [1].

FOOTNOTE [1]. This formal condition, obviously, gives no real advantages to player A. ENDFOOTNOTE.

The development of game in time we will represent that consist of a series of consecutive stages or "courses". Course in the theory of games is called selection of one of those provided by the rules of the game of actions and its realization.

Courses occur personal and random. By personal course is called conscious selection by player of one of the possible versions of actions and his realization (example - any course in checkered game). Chance move is called the selection from a series of possibilities, carried out not by the decision of player, but by any mechanism of random sampling (coin-tossing, the selection of map/chart from the shuffled block, etc.). For each chance move of the rule of game, is determined the probability distribution of possible issues.

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Some games consist only of chance moves (the so-called purely games of chance) either only of chance moves (the so-called purely games of chance) or only of personal courses (chess, cartridges). The majority of card games contains both personal and chance moves.

The theory of games is occupied by the analysis only of those games which contain personal courses; its problem - to give indications to players when selecting of their personal courses, i.e., to recommend to them determined "strategies".

Player's strategy is called the set of the rules, which

determine the selection of the version of actions with this player's each personal course depending on the situation, which established in the process of game.

The concept of strategy - one of basic in the theory of games; let us pause at it in somewhat more detail. Usually, taking part in game, player does not follow some rigid, fixed/recorded rules: selection (solution) with each personal course is accepted by it in the course of game, depending on the establishing concrete/specific/actual situation. However, theoretically matter will not be changed, if we visualize that all these solutions are accepted by player previously (" if it is summed certain situation, I I will act such a one"). In the principle (if not virtually) this is possible for any game. If this system of solutions will be accepted, this will mean that the player selected specific strategy. Now he can and not participate in game personally, but replace his participation by the list of the rules which for it it will apply the disinterested face (judge). Strategy can be also assigned to machine-automat in the form of the program (precisely so they play the chess electronic computers).

Depending on the number of possible strategies of game, are divided into "final" and "infinite".

Game is called final, if at each player has only finite number of strategies, and infinite, if at least in one of the players is an infinite number of strategies.

The target/purpose of the theory of games is the consumption/production/generation of recommendations for players's reasonable behavior in conflicting situation, i.e., the determination "optimal strategy" for each of them.

The optimal strategy of player is called such strategy which during the multiple repetition of game ensures to this player a maximally possible average gain (or, which is the same thing, smallest possible average loss). When selecting of this strategy the basis of reasonings is the assumption that the enemy at least poppy is reasonable as and we themselves, and makes everything in order to prevent us to attain its goal.

In the theory of games, all recommendations are developed proceeding precisely from these principles; consequently, in it are not considered errors and players's errors, the unavoidable in each conflicting situation, or cell/elements of ardor and risk.

The theory of games as any mathematical model of complex phenomenon, has its limitations. Most important of them is the fact

that the gain artificially is reduced to one-single number.

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In the majority of conflicting situations when selecting of reasonable strategy, it is necessary to take into attention not one, but several numerical parameters - indices of efficiency. Strategy, optimum for one index, will be not necessarily optimum for others. Realizing these limitations and therefore without adhering to blindly the recommendations, obtained by play methods, it is possible all the same reasonable to use a mathematical apparatus of the theory of games for consumption/production/generation, if not in the accuracy of optimum, then, in any case of "acceptable" strategy.

3. Payoff matrix.

Let us consider the final game in which player A (" we") has m of strategies, and player B ("enemy") - n of strategies. This game is called game m x n. Let us designate our strategies A_1, A_2, ... A_m; strategy of enemy - B_1, B_2, ..., B_n. Let us assume that each side selected specific strategy: we selected A_i, enemy - B_j. If game consists only of personal courses, then the selection of strategies A_i, B_j uniquely determines the issue of game - our gain (positive or negative); let us designate it a_{ij}.

If game contains besides personal chance moves, then gain with the pair of strategies A_i, B_j there is a value random, that depends on the issues of all chance moves. In this case the natural estimation of the expected gain was the mathematical expectation of random gain. We will designate one and the same sign a_{ij} both gain itself (in game without chance moves) and its mathematical expectation (in game with chance moves).

Let us assume that to us are known the values a_{ij} with each pair of strategies. These values can be registered in the form of rectangular array (matrix/die) whose rows correspond to our strategies (A_i), and columns - to strategies of enemy (B_j):

A_i \ B_j	B_1	B_2	...	B_n
A_1	a_{11}	a_{12}	...	a_{1n}
A_2	a_{21}	a_{22}	...	a_{2n}
...	...	...	...	...
A_m	a_{m1}	a_{m2}	...	a_{mn}

This table is called payoff matrix or it is simple by the matrix/die of game.

Let us note that the construction of payoff matrix, especially for games with a large quantity of strategies, can by itself represent very complex problem.

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For example, for a checkered game the number of possible strategies so is great, that construction of payoff matrix (even with the enlistment of computers) is thus far virtually unrealizable. However, in principle any final game can be given to matrix form.

Let us examine several elementary examples of games let us construct for them payoff matrices.

Example 1. Game "search",

There are two players A and B; player A hides, in B it he seeks. In order A, there are two refuges (I and II), any of which it can select at its discretion. The conditions of game are such: if B finds A in that refuge where A hid, then A pays to it penalty 1 rubles; if B does not find A (i.e. it will seek in other refuge), then it itself must pay A of the same staff. It is required to construct payoff matrix.

Solution. Game consists in all of two courses, both - personal. Of us (A) two strategies: A_1 - to hide in refuge I, A_2 - to hide in refuge II.

Of the enemy (B) also two strategies: B_1 - to seek in refuge I, B_2 - to seek in refuge II.

Before us - game 2 x 2. Its matrix/die takes the form:

A_i \ B_j	B_1	B_2
A_1	−1	1
A_2	1	−1

Based on the example of this game, as it not is elementary, it is possible to explain to itself some important ideas of the theory of games.

Let us assume first that this game is implemented only one time (is played only "party/batch"). Then, obviously, there is no sense to speak about the advantages of one or the other strategies - each of the players it can with the equal basis/base to take any of them.

However, during the multiple repetition of game, position varies.

It is real/actual let us assume that we (player A) selected some strategy (let us say, that A_1) we adhere to it. Then, already according to the results of the first several party/batches, enemy is conjectured about our strategy, it will begin always to seek in refuge 1 and to win. The same will be, if we select strategy A_2. To us to clearly disadvantageous adhere to one some strategy; in order not to render/show in loss, we must alternate them.

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However, if we are alternate refuges I and II in some specific sequence (let us say, that through one party/batch), the enemy also will be conjectured about this and will answer by the worst for us form. It is obvious, the reliable method, which guarantees us from accurate loss, it will be such organization of selection in each party/batch when we themselves it do not know in advance. For example, it is possible to throw coin, and, if falls coat of arms, to select refuge I, if tail - refuge II.

The sad position in which render/showed player A (in order not to lose, to choose refuge randomly), obviously, it is been inherent not only in it, but also to his enemy B, for whom were valid all

reasonings given above. The op'‸mal strategy of each proves to be "mixed" strategy, in which player's two possible strategies are alternated randomly, with identical probabilities.

Thus, we via the intuitive reasonings arrived at one of the essential concepts of the theory of games - to the concept of the mixed strategy - i.e. such, in which separate "pure/clean" strategies are alternated randomly with some probabilities. In this example from the considerations of symmetry, it is clear that strategies A_1 and A_2 must be accepted itself with identical probabilities; in more complex examples the solution can be by no means trivial.

Example 2. Game "three fingers".

Players A and B it is simultaneous and independently of each other show one, two or three finger/pins. Gain or loss solves the total number of shown finger/pins. Gain (in rubles) is equal to this number; if it even - wins A, and B to it pays; if odd - vice versa. It is required to construct payoff matrix.

Solution. Of each player on three strategy: to show one, two or three finger/pins. The matrix/die of game 3 x 3 takes the form:

B_j / A_i	B_1	B_2	B_3
A_1	2	—3	4
A_2	—3	4	—5
A_3	4	—5	6

We analyze situation. It is obvious, to any our strategy the enemy can answer by the worst for us form. For example if we select A_1, it answers us B_2, and we will lose 3 rubles. Strategy A_2, it us will answer B_3, and we will lose 5 rubles; to strategy A_3 - B_2, and we again lose 5 rubles. It is obvious, certain advantage has strategy A_1 (with it loss is minimal), but also it for us is clearly unfavorable, since always it leads to loss.

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However, let us try to stop to the point of the second player (B). Its position also not from bright. If it selects B_1, we will answer it A_3, and it will return to us 4 rubles; if B_2 - we will answer A_2 and we will again obtain 4 rubles; also on B_3 of us answer/response A_3, which leads to the even worse result: B it will lose 6 rubles.

It leaves, game is unfavorable neither that nor other of the players: each of them, after selecting some specific strategy, is condemned to loss ! This suggests, that also here output/yield - in the application/use of mixed strategies; it is real/actual, so it there is, but in this example matter is not as simply as in previous, and in order to find the optimal strategies of sides, it is necessary to learn to solve games. Subsequently we will return to this example and will find its solution.

Example 3. Game is "armament and aircraft". Available are three forms of the armament: A_1, A_2, A_3; of enemy - three forms of the aircraft: B_1, B_2, B_3. Our problem - to strike aircraft; the problem of the enemy - to preserve him that nonafflicted. Our personal course - selection of the type of armament; the personal course of the enemy - selection of aircraft for combat operations. In this game there is even a chance move - application of armament. By armament A_1 aircraft B_1, B_2, B_3 are surprised in accordance with probabilities 0.5, 0.6, 0.8; by armament A_2 = with probabilities 0.9, 0.7, 0.8; by armament A_3 - with probabilities 0.7, 0.5, 0.6. To construct the matrix/die of game and to analyze situation.

Solution. The matrix/die of game 3x3 takes the form:

B_j / A_i	B_1	B_2	B_3
A_1	0,5	0,6	0,8
A_2	0,9	0,7	0,8
A_3	0,7	0,5	0,6

where the gain - kill probability of the aircraft (we attempt it to maximize, and enemy - to minimize).

Above this game it is worth thinking, since it possesses some special properties, imperceptible on first glance.

Let us become first to the point of player A and will sort out one for another all his strategies. A_1 the enemy will answer us B_1, and we will win 0.5; on A_2 - B_2, and we will win 0.7; on A_3 - again B_2, and we will win 0.5; to A_2-B_2, and we will win 0.7; on $_3$ - again B_2, and we will win 0.5. It is obvious, certain advantage above others has strategy A_2 - with it we let us win more, namely 0.7.

Let us become now to the point of enemy; we will not forget, that it wishes to return a little less ! Let it choose B_1 - we answer it A_2, and it gives up 0.9; B_2 we answer it A_2, and it gives up 0.7;

on B_3 - A_3, and it gives up 0.8. It is logical, it will prefer B_2 in order to return only to 0.7.

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We see that in this example of strategy A_2 and B_2 with gain 0.7 they are most advantageous immediately for both sides; to player A it is more advantageous anything to choose strategy A_2, to player B - strategy B_2, Ci the maximum gain A coincides with the minimum loss B. Is reached as if position of equilibrium: if A selects strategy A_2, then B cannot find the best output/yield, than B_2, and vice versa: if B it selects strategy B_2, then A it cannot find the best output/yield, than A_2.

Subsequently we will see, that the pair of strategies, which possess this property, they are the optimal strategies of sides and form the so-called solution of game.

4. Lower and upper pure value. Minimax principle.

Let us examine game m x n with the matrix/die

$A_i \backslash B_j$	B_1	B_2	...	B_n
A_1	a_{11}	a_{12}	...	a_{1n}
A_2	a_{21}	a_{22}	...	a_{2n}
...	...	...	...	...
A_m	a_{m1}	a_{m2}	...	a_{mn}

Letter i let us designate the number of our strategy, by letter j - number of strategy of enemy.

Let us reject/throw a question concerning mixed strategies and will examine thus far only pure/clean. Let us assign the mission: to determine best among our strategies A_1, A_2, ..., A_m. We analyze consecutively each of them, beginning with A_1 and ending A_m. We analyze consecutively each of them, beginning with A_1 and ending A_m. Choosing A_i, we they must calculate, that the enemy will answer it by that of strategies B_j, for which our gain it is minimal. Let us find minimum from numbers a_{ij} in the i row and will designate it α_i:

$$\alpha_i = \min_j a_{ij} \qquad (4.1)$$

(sign min it designates the minimum value of this parameter at all possible j).

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Let us write out numbers α_i (minimums of rows) next to matrix/die to the right in the form of the additional chair:

B_j \ A_i	B_1	B_2	...	B_n	α_i
A_1	a_{11}	a_{12}	...	a_{1n}	α_1
A_2	a_{21}	a_{22}	...	a_{2n}	α_2
...	...	...	...	...	...
A_m	a_{m1}	a_{m2}	...	a_{mn}	α_m
β_j	β_1	β_2	...	β_n	

(Минимумы строк) (1)

(Максимумы столбцов) (2)

Key: (1). (minimums of rows). (2). (Maximums of columns).

Choosing some strategy A_i, we they must rely on the fact that as a result of the reasonable actions of enemy we will win only α_i. It is logical, functioning most carefully (i.e. avoiding any risk), we must prefer to other TU strategy, for which the number α_i is maximal. Let us designate this maximum value α:

$$\alpha = \max_i \alpha_i,$$

or taking into account formula (4.1),

$$\alpha = \max_i \min_j \alpha_{ij}. \qquad (4.3)$$

Value α is called lower worth of game, otherwise - by maximin

gain or maximin. Player's that strategy A which corresponds to maximin α it is called maximin strategy.

It is obvious, if we are adhere to maximin strategy, then to us with any behavior of the enemy is guaranteed gain, in any case, not smaller α. Therefore value α is called "lower worth of game". This - that guaranteed minimum which we can to ourselves provide, adhering to its most careful (" reinsurance") strategy.

It is obvious, analogous reasoning can be led, also, for enemy B. It is of interest to convert our gain into the minimum; that means it must look over all its strategies, selecting for each of them the maximum value of gain.

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Let us write out below matrix/die (4.2) maximum values a_{ij} on the chairs:

$$\beta_j = \max_i a_{ij}$$

let us find them them minimum:

$$\beta = \min \beta_j$$

or

$$\beta = \min_j \max_i a_{ij}. \tag{4.4}$$

Value β is called upper pure value, otherwise minimax gain or minimax. Corresponding to gain β strategy of the enemy is called his minimax strategy. Adhering to his most careful minimax strategy, enemy is guaranteed, that in any event he will lose not more than β.

The principle of precaution, which dictates to players the selection of corresponding strategies (maximin and minimax), is in the theory of games basic and is called the minimax principle. It escape/ensues from assumption about the soundness of each player, who attempts to achieve the target/purpose, opposite to the target/purpose of enemy. Most "careful" maximin and minimax strategies frequently designate by common/general/total term "minimax strategies".

Let us determine lower and upper pure values, and also minimax strategies, for three examples, examined in the previous paragraph.

Example 1. (Game is "search"). Determining the minimums of rows α_i and the maximums of columns β_j, we will obtain

B_j \ A_i	B_1	B_2	α_i
A_1	−1	+1	−1
A_2	+1	−1	−1
β_j	1	1	

Since values α_i and β_j are constant and equal to with respect -1 and +1, lower and upper pure values are also equal to - 1 and +1:

$$\alpha = -1, \beta = +1.$$

Player's any strategy A is him maximin, and player B - by his minimax strategy. Conclusion/derivation it is trivial: adhering to any of his strategies, player A can guarantee, that he will lose not more than 1 rub.; the same can guarantee player B with his any strategy.

Example 2. (Game "three fingers"). Writing out the minimums of rows and the maximums of chairs, let us find lower worth of game α = -3 and upper β=4 (are isolated in table by bold type).

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Our maximum strategy A_1 (applying it schematically, we guarantee,

B_j \ A_i	A_1	A_2	A_3	α_i
A_1	0,5	0,6	0,8	0,5
A_2	0,9	0,7	0,8	**0,7**
A_3	0,7	0,5	0,6	0,5
β_j	0,9	**0,7**	0,8	

that let us win not less than -3, i.e., let us lose not more than 3). Minimax strategy of enemy - any of strategies B_1 and B_2; applying them systematically, it can guarantee, that it will not return more than 4. If we step back from our maximin strategy (for example, let us select A_2); the enemy it can us "punish" for this, after using B_3 and after reducing our gain k-5; equally and the departure of the enemy from his minimax strategy can be "punished" by an increase in its loss to 6.

Let us focus attention on the fact that minimax strategies in this case are not stable. It is real/actual, let, for example, the enemy select one of his minimax strategies B_1 and he adheres to her. After learning about this, we will pass to strategy A_3 and will win 4. On this enemy will answer by strategy B_2 and it will win 5; this we, in turn, will answer by strategy A_2 and will win 4, and so forth. Thus, the position in which both player use their minimax strategies, is unstable and can be broken by the acted information about strategy which applies the contrary side. However, this instability is observed not always; of this, we will be convinced based on following example.

Example 3. (Game is "armament and aircraft"). We determine the minimums of rows and the maximums of the chairs:

A_i \ B_j	B_1	B_2	B_3	β_i
A_1	2	−3	4	−3
A_2	−3	4	−5	−5
A_3	[illegible]	−5	6	−5
β_j	4	4	6	

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In this case lower worth of game is equal to upper:

$$\alpha = \beta = 0{,}7$$

Minimax strategies A_2 and B_2 are stable: if one of the players adheres to its minimax (maximin) strategy, then another player in any way cannot improve his position, stepping back from its.

Thus, we see that there are games for which lower value is equal to upper:

$$\alpha = \beta.$$

These games occupy the special place in the theory of games and they are called games with saddle point. In the matrix/die of this game, there is a cell/element, which is simultaneously minimum in its

row and maximum in its chair; this cell/element is called "saddle point" (by analogy with saddle point on the surface where it is reached the minimum on one coordinate and the maximum on another).

The common/general/total value of lower and upper pure value

$$\alpha = \beta = \nu$$

is called pure/clean worth of game.

To saddle point corresponds the pair of minimax strategies; these strategies are called optimum, and their set - by a solution of game. The solution of game possesses the following property: if one of the players adheres to its optimal strategy, then for another it cannot be advantageous to differ from its optimum (this deviation either will leave position constant/invariable or it will impair it).

Actually, let in game with saddle point player A holds its optimal strategy, and player B - by its. As long as this so - gain remains constant and equal to worth of game v. Now let us assume that B allowed deviation from its optimal strategy. Since cell/element is minimum in its row, this deviation cannot be advantageous for B; equally and for A, if B adheres to its optimal strategy, there cannot be profitably deviation from its.

We see that for a game with saddle point minimax strategies

possess stability. The pair of optimum strategies in the game with saddle point is as if position of equilibrium: deviation from the optimal strategy causes such change in the gain which disadvantageously for the deviating player forces him to return to its optimal strategy.

Pure/clean worth of game in game with saddle point is that value of gain which in game against reasonable enemy player A cannot increase, but player B - to decrease.

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Let us note that in payoff matrix there can be not one saddle point, but several. For example, in matrix/die is six saddle points,

A_i \ B_j	B_1	B_2	B_3	B_4	B_5	α_i
A_1	2	2	1	1	2	1
A_2	0	1	1	1	1	0
A_3	1	1	1	1	2	1
A_4	1	2	1	1	2	1
β_j	2	2	1	1	2	

with the common/general/total value of gain $\alpha = \beta = \nu = 1$ and the corresponding pairs of the optimal strategies: A_1B_3, A_1B_4, A_3B_3, A_3B_4, A_4B_3, A_4B_4. It is not difficult to demonstrate (we this make will not) that if in the matrix/die of game several saddle points, then they all give one and the same value of gain.

Example. Side A (air defense weapon defends from the air raid the section of territory, disposing of two instruments No 1 and No 2 the zones of action of which S_1, S_2 do not overlap (Fig. 9.1). Each instrument can fire only aircraft, passing through its zone actions, but for this it must previously (prior to the entrance of target/purpose into zone) follow it and develop sighting data. If target/purpose is fired, it is surprised with probability p = 1. Side B has available two aircraft each of which can be directed to any zone B the torque/moment when side A realizes the target assignment (it assigns, to which instrument on which target/purpose of shooting), moving target aircraft No 1 is directed to the zone of action S_1 of instrument No 1, and target No 2 - into the zone of action S_2 of instrument No 2. However, after making of decision by target assignment, each target/purpose can maneuver, after using "misleading maneuver" (see broken pointers in Fig. 9.1). Problem of side A - to convert to maximum, and sides B - to convert into the minimum the number of affected target/purposes. To find the solution of game (the optimal strategies of sides).

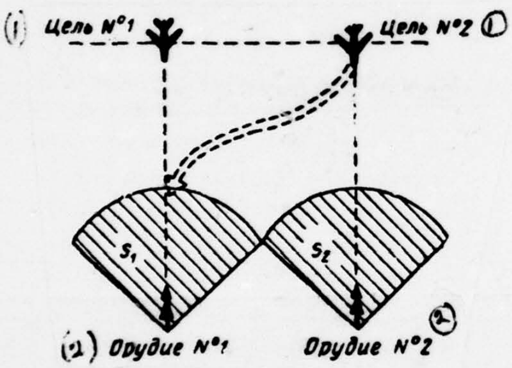

Fig. 9.1. Key: (1). Target/purpose. (2). Instrument.

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Solution. Of side A (air defense weapon four possible strategies

A_1 - each weapon follows after directed for its zone target.

A_2 - instrument they track a target "cross- crosswise" (each - after the target/purpose, which is directed toward neighbor),

A_3 - both instruments they track a target No 1.

A_4 - both instruments they follow target No 2.

Of side B (target aircraft) also four strategies:

B_1 - both target/purposes do not vary direction,

B_2 - both target/purposes applies misleading maneuver.

B_3 - the first target/purpose is applied misleading maneuver, but the second no,

B_4 - the second target/purpose is applied misleading maneuver, but the first no.

Is obtained game 4 x 4 whose matrix/die is given in the table:

A_i \ B_j	B_1 ↓↓	B_2 ⤫	B_3 ↘↓	B_4 ↓↙	α_i
A_1 ↑↑	2	0	1	1	0
A_2 ⤫	0	2	1	1	0
A_3 ↑↖	1	1	1	1	1
A_4 ↗↑	1	1	1	1	1
β_j	2	2	1	1	

Finding the minimums of rows and the maximums of chairs, we are convinced that the lower worth of game is equal to upper pure value: $\alpha=\beta-\nu=1$; that means game it has saddle point and solution in the pure strategies, which leads to pure/clean worth of game $\nu=1$. In this case of saddle points not one, but whole four each of them corresponds the pair of the optimal strategies, which gives the solution of game. Worth of game $\nu=1$ means that with the optimum behavior of sides the aircraft will unavoidably lose one aircraft, and any contrivances will aid them to lose less, but to the air defense weapons - to bring down are more than one aircraft. Is reached this state of the equilibrium when both sides use their optimal strategies: instruments follow both one and the same aircraft (by any), and aircraft are directed after target assignment for one and the same zone (any).

The class of the games, which have saddle point, is very interesting both from theoretical and from the practical point. To it belong, in particular, all the so-called "perfect information games".

Perfect information game is called such game in which each player with each personal course knows the results of all previous courses - both personal and random. As examples of games with complete information can serve: cartridges, chess, known game the "crosses and zeroes", etc.

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In the theory of games, is proven, that each perfect information game has saddle point and consequently, solution in the pure strategies. In other words, in each perfect information game there is a pair of the optimal strategies of each side, which gives the stable gain, equal to pure/clean worth of game. If perfect information game consists only of personal courses, then during application/use by each side of its optimal strategy game must end with always completely determined by the issue, equal to worth of game v.

As an example let us give following perfect information game. Two Players they alternately put identical coins to circular table, choosing arbitrarily the position of the coin (mutual overlap of

coins is not allow/assumed). It wins the one who will place the last/latter coin (when places for others no longer will remain). It is not difficult to ascertain that the issue of this game is decided beforehand, and there exists specific strategy, which ensures reliable gain to that of the players, who puts coin by the first. Namely, he must for the first time place coin to the center of table, and further on each course of enemy to answer symmetrical course. It is obvious, no matter how behaved enemy, to him not to avoid loss. Therefore game makes sense only for those, who do not know its solution. In exactly the same manner matter is with the chess and other perfect information games; any of these games possesses saddle point and, which means, that by the solution, which indicates to each player his optimal strategy, so that game makes sense only, while is unknown solution. The solution of checkered game found (and in the foreseeable future scarcely whether it will be found) only because the number of strategies (combinations of courses) in chess is too great so that it is possible former to construct the payoff matrix and to find in it saddle point.

5. Solution of game in mixed strategies.

Among the final games, which are of practical use, not too frequently meet the games saddle point; more typical is the case when lower and upper pure values are different. Analyzing the matrix/dies

of such games, we arrived at the conclusion that if we to each player let the selection of only one pure strategy, then taking into account reasonable enemy this selection must be determined by the minimax principle. With this player A guarantees to itself a win, equal to lower worth of game α. Does arise the question: it is not possible whether to guarantee the gain, larger than α, if we apply not a one-only, "pure/clean" strategy, but to alternate randomly several strategies? Such strategies, which consist of the random rotation of the pure strategies, are called in the theory of games displaced. With the use of mixed strategy before each party/batch of game, is released into course some mechanism of random sampling (coin-tossing, the die or calculation by the machine of random number from 0 to 1), that ensures the appearance of each strategy with certain probability, and then is accepted that strategy, to which fell the toss.

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Mixed strategies represent mathematical model of variable, by bending the tactics, with which the enemy does not know, and it cannot learn previously, with which situation for it it is necessary to be met. This random rotation of methods (of course, without the clearly determined probabilities) they frequently use in card games.

Let us introduce special designation for mixed strategies. Let there be the game И, in which of us (A) m of strategies: $A_1, A_2, \ldots, A_m$, and of enemy (B) - n strategies: $B_1, B_2, \ldots, B_n$. Let us designate

$$S_A = (p_1, p_2, \ldots, p_m)$$

our mixed strategy in which strategies $A_1, A_2, \ldots, A_m$ are applied with probabilities $p_1, p_2, \ldots, p_m$, moreover $p_1+p_2+\ldots+p_m=1$.

Analogous designation for the mixed strategy of the enemy will be

$$S_B = (q, q_2, \ldots, q_n),$$

where $q_1+q_2+\dots+q_n=1$.

It is obvious, each pure/clean strategy is a special case mixed: all strategies, except datum, have probabilities, equal to zero, and datum - to one.

It proves to be, if we allow not only pure/clean, but also mixed strategy, then it is possible for each final game to find solution, i.e., the pair of the stable optimal strategies of the players.

The solution of game is called the pair of optimal strategies S_A^*, S_B^*, in the general case mixed, which possess the following property: if one of the players holds his optimal strategy, then to another it cannot be profitable to step back from its.

The gain, which corresponds to solution, is called worth of game; we will (as is earlier - pure/clean value) designate it ν.

There is the so-called fundamental theorem of the theory of games, which consists of following.

Each final game has at least one solution, possibly, in the

range of mixed strategies.

We will not be stopped on the strict proof of this theorem, especially because subsequently the existence of the solution of game will be sufficiently obviously from other considerations.

From fundamental theorem it follows that each final game has a value. Worth of game v always lie/rests between lower worth of game α and upper pure value β:

$$\alpha \leqslant v \leqslant \beta.$$

It is real/actual, α there is the maximum guaranteed gain, which we can to ourselves ensure, applying its only pure strategies. Since mixed strategies contain as a special case everything pure/clean, then, allow/assuming besides pure/clean even mixed strategies, we, in any case, will not impair our possibilities; that means

$$v \geqslant \alpha.$$

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It is analogous, examining the possibilities of enemy, let us demonstrate that

$$v \leqslant \beta,$$

whence $\alpha \leqslant v \leqslant \beta$.

Let us assume that in game m x n we have is found solution, which consists of two optimal strategies:

$$S_A^* = (p_1, p_2, \ldots, p_m); \quad S_B^* = (q_1, q_2, \ldots, q_n).$$

In the general case, some of numbers $p_1, p_2, \ldots, p_m; q_1, q_2, \ldots, q_n$ can be equal to zero, i.e., not all strategies, available to player, enter in his optimum mixed strategy. Let us call player's active strategies those that enter in his optimum mixed strategy with different from zero probabilities. For the solution of games vital importance has the following theorem about active strategies.

If one of the players holds his optimum mixed strategy, then gain remains constant/invariable and equal to worth of game v, regardless of the fact the fact that makes another player, if only that does not fall outside the apparitors of its active strategies (i.e., is used any of them in pure form or are mixed them in any proportions).

Let us demonstrate this theorem. Let there be the solution of game m x n in mixed strategies, in which some strategies are active, but others no. Let us index strategies that so that active would be first k of player's strategies A and first l of player's strategies

B. Solution will take the form:

$$S_A^* = (p_1,\ p_2,\ \ldots,\ p_k,\ 0,\ \ldots,\ 0), \qquad (p_1 + p_2 + \ldots + p_k = 1);$$
$$S_B^* = (q_1,\ q_2,\ \ldots,\ q_l,\ 0,\ \ldots,\ 0), \qquad (q_1 + q_2 + \ldots + q_l = 1),$$

and its application/use leads to the gain, equal to worth of game v.

It is claimed that if we (A) will adhere to our of strategies S_A^*, the enemy (B) can apply his strategies B_1, $B_2, \ldots, B_l$ (but not $B_{l+1}, \ldots, B_n$) in any proportions; the gain in this case remains constant and equal to v.

Let us designate $v_1, v_2, \ldots, v_l$ the gain, forming, if we use optimal strategy S_A^*, and enemy - by pure strategies $B_1, B_2, \ldots, B_l$. From the determination of the solution of game, it follows that one-sided deviation of enemy from his optimal strategy cannot be to it profitably; therefore

$$v_1 \geqslant v; \quad v_2 \geqslant v; \ \ldots; \ v_l \geqslant v.$$

Let us look, can at least one of values $v_1, v_2, \ldots, v_l$ turn out to be actually more than v. It turns out no. Actually, is expressed gain v with optimal strategies S_A^*, S_B^* through gains $v_1, v_2, \ldots, v_l$.

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Since in mixed strategy S_B^* pure strategies $B_1, B_2, \ldots, B_l$ are applied with probabilities $q_1, q_2, \ldots, q_l$, then average gain will be:

$$v = v_1 q_1 + v_2 q_2 + \ldots + v_l q_l = \sum_{j=1}^{l} v_j q_j, \qquad (5.1)$$

moreover

$$q_1 + q_2 + \ldots + q_l = 1.$$

It is obvious that if of values $v_1, v_2, \ldots, v_l$ at least one was more than v, then their also mean suspended value (5.1) would be more than v, which contradicts condition. Thus, is demonstrated the theorem, which we will use extensively during the solution of games.

6. Simplification in the games.

If game m x n does not have saddle point, the finding of its solution, especially with large m and n, represents by itself sufficiently laborious problem. Sometimes this problem can be simplified, if we preliminarily" reduce" game, i.e., to reduce the number of strategies by the deletion of some excessive.

Excessive strategies are of two kinds: duplicating and knowingly unfavorable.

Let us consiler, for example game, *И*, with the matrix/die:

$A_i \backslash B_j$	B_1	B_2	B_3	B_4
A_1	1	2	4	3
A_2	0	2	3	2
A_3	1	2	4	3
A_4	4	3	1	0

From matrix/die it is evident that strategy A_3 in accuracy repeats ("duplicate/backs up/reinforce") strategy A_1; therefore any of these two strategies can be crossed out. Further, equate/comparing piecemeal rows A_1 and A_2, we see that all row elements A_2 are less (or are equal) equivalent components of row A_1. That means that strategy A_2 for us, that desire to win, is knowingly unfavorable. Deleting A_3 and A_2, let us lead matrix/die to the simpler form:

$A_i \backslash B_j$	B_1	B_2	B_3	B_4
A_1	1	2	4	3
A_4	4	3	1	0

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We further note that for an enemy strategy B_3 is knowingly unfavorable; we delete it, and matrix/die is given to the form:

A_i \ B_j	B_1	B_2	B_4
A_1	1	2	3
A_4	4	3	0

Thus, game 4 x 4 is reduced to game 2 x 3.

Sometimes it is possible to simplify game by artificial introduction instead of the pure strategies - mixed. Let there be, for example, the game 3 x 4 with the matrix/die:

A_i \ B_j	B_1	B_2	B_3	B_4
A_1	0	5	5	2
A_2	5	0	2	5
A_3	5	5	1	1

Examining matrix/die, we note that, in view of symmetry of column elements B_1 and B_2; B_3 and B_4, and also rows A_1 and A_2, these strategies, if they enter in solution, then only with the identical probabilities: $p_1 = p_2$, $q_1 = q_2$, $q_3 = q_4$. Hence appears the idea: to previously join strategies B_1 and B_2 into one mixed strategy B_{12}, which consists half of B_1, half of B_2; so to act with strategies B_3 and B_4, i.e., to join them one mixed strategy B_{34}, in which B_3 and B_4 they enter with identical probabilities 1/2. We lead matrix/die to the form:

A_i \ B_j	B_{12}	B_{34}
A_1	2,5	3,5
A_2	2,5	3,5
A_3	5	1

Now it is evident that if the enemy uses strategies B_{12}, B_{34}, strategies A_1 and A_2 duplicate/back up/reinforce each other; deleting any of them (or joining A_1 and A_2 into one A_{12}), we lead matrix/die to form 2 x 2:

A_i \ B_j	B_{12}	B_{34}
A_{12}	2,5	3,5
A_3	5	1

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Thus, game 3 x 4 is reduced to game 2 x 2.

Beginning the solution of any game m x n, it is necessary to first of all fulfill the following procedures:

- to look, no whether in the matrix/die of saddle point: if be,

solution is already found;

- if there is no saddle point, to compare between themselves piecemeal columns and rows for purpose of the deletion of duplicating and knowingly unfavorable strategies;

- to look, it is not possible to decrease the number of strategies way replacement of some groups or pure/clean - those mixed.

7. *Game* 2 x 2.

The simplest case of final game is game 2 x 2, where of each player two strategies. Let us consider game 2 x 2 with the matrix/die:

A_i \ B_j	B_1	B_1
A_1	a_{11}	a_{12}
A_2	a_{21}	a_{22}

Here can be met two cases:

1) game has saddle point;

2) game does not have saddle point.

In the first case solution is obvious: this - the pair of strategies of those intersecting at saddle point. It is not difficult to demonstrate that if the game 2 x 2 has saddle point, then in this game always any of strategies there can be rejected as knowingly unfavorable or duplicating. We will not this prove. Let us let to reader to demonstrate this position or to be convinced of his validity on a series of the arbitrarily selected examples.

Let us consider the second case: let us assume that in matrix/die 2 x 2 there are no saddle points. In this case, lower worth of game is not equal to upper $\alpha \neq \beta$. Solution must be in mixed strategies. let us find this solution, i.e., the pair of the optimum mixed strategies:

$$S_A^* = (p_1, p_2); \quad S_B^* = (q_1, q_2).$$

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Let us first determine optimum mixed strategy S_A^*. According to the theorem about active strategies (see § 5), if we are adhere to this strategy, then, independent of the modus operandi of enemy (if it only it does not exceed the limits of its active strategies), gain will remain equal to worth of game v. In game 2 x 2, both strategies

of enemy are active (otherwise game would have saddle point). That means if we adhere to our optimal strategy $S_A^* = (p_1, p_2)$, then enemy can, without varying gain, to apply any of its pure strategies. We hence have two equations:

$$\left.\begin{aligned} a_{11}p_1 + a_{21}p_2 = v, \\ a_{12}p_1 + a_{22}p_2 = v, \end{aligned}\right\} \quad (7.1)$$

from which, taking into account condition $p_1 + p_2 = 1$ we will obtain:

$$\left.\begin{aligned} p_1 &= \frac{a_{22} - a_{21}}{a_{11} + a_{22} - a_{12} - a_{21}}, \\ p_2 &= 1 - p_1 = \frac{a_{11} - a_{12}}{a_{11} + a_{22} - a_{12} - a_{21}}. \end{aligned}\right\} \quad (7.2)$$

Worth of game v let us find, substituting the value p_1, p_2 in any of equations (7.1):

$$v = \frac{a_{22}a_{11} - a_{12}a_{21}}{a_{11} + a_{22} - a_{12} - a_{21}}. \quad (7.3)$$

analogously it is located the optimal strategy of the enemy:

$$S_B^* = (q_1, q_2)$$

from the equations

$$\left.\begin{aligned} a_{11}q_1 + a_{12}q_2 = v, \\ a_{21}q_1 + a_{22}q_2 = v, \end{aligned}\right\} \quad (7.4)$$

whence

$$\begin{aligned} q_1 &= \frac{a_{22} - a_{12}}{a_{11} + a_{22} - a_{12} - a_{21}}, \\ q_2 &= 1 - q_1. \end{aligned} \quad (7.5)$$

Example 1. To find the solution of game "search" (see example 1 of §2).

Solution. Game 2 x 2 with the matrix/die

B_j / A_i	B_1	B_2
A_1	-1	1
A_2	1	-1

does not have saddle point: $\alpha = -1$, $\beta = +1$. We seek solution in mixed strategies.

On formulas (7.2), (7.3), (7.5) we obtain:

$$p_1 = {}^1/_2;\ p_2 = {}^1/_2;\ \ \nu = 0;\ q_1 = {}^1/_2;\ q_2 = {}^1/_2;$$
$$S_A{}^* = ({}^1/_2,\ {}^1/_2);\ \ S_B{}^* = ({}^1/_2,\ {}^1/_2).$$

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Consequently, the optimal strategy of each player lies in the fact that, randomly alternating of its pure strategies, using each of them with probability 1/2; in this case, average gain will be equal to zero (this conclusion/derivation already have obtained we from intuitive considerations). In a following example we will consider the game whose solution is not so/such obvious.

Example 2. Game "two bombers and destroyer".

Side A sends into region of the location of enemy B two bombers I and II; I flies from the front, II- from behind. One of the bombers (it is previously unknown, which) must carry bomb; another implements only the function of tracking. In enemy region the bombers are under attack of the destroyer of side B (Fig. 9.2). Both bomber are armed by guns. If destroyer attacks rear bomber, then on it conduct the fire/light of the gun only of this bomber, that damage destroyer with probability 0.3. But if destroyer attacks front bomber, on it they conduct the fire/light of the gun both front/leading and rear bomber; together they strike it with the probability

$$1-(1-0{,}3)^2=0{,}51.$$

If destroyer is not biased/beaten by the return fire of bombers, then it strikes selected by it target with probability 0.8.

Problem of bombers - to report bomb to target/purpose; the problem of destroyer - to prevent this.

It is required to find optimum strategies of the sides:

- For side A - which bomber to do by a carrier?

- For side B - which bomber to attack?

Solution. Let us comprise the matrix/die of games, for which will find average gain during each combination of strategies. Gain - probability of the nondestruction of carrier.

1. A_1B_1 - carrier 1, is attacked 1. Carrier will not be struck, if bombers will bring down destroyer, or if they it do not bring down, but also it will not strike its target/purpose. Probability that both bomber will together strike fighter, is equal to 0.51; therefore

$$a_{11} = 0{,}51 + (1 - 0{,}51)(1 - 0{,}8) = 0{,}608.$$

2. A_2B_1 - carrier II, is attacked I;

$$a_{21} = 1.$$

3. A_1B_2 - carrier I, are attacked II;

$$a_{12} = 1.$$

4. A_2B_2 - carrier II, are attacked II;

$$a_{22} = 0{,}3 + 0{,}7 \cdot 0{,}2 = 0{,}44.$$

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Matrix/die of game with additional column and the row:

A_i \ B_j	B_1	B_2	α_i
A_1	0,608	1	0,608
A_2	1	0,44	0,44
β_j	1	1	

Lower - the worth of game α = 0.608, upper β = 1. Game does not have saddle point; solution is achieved in mixed strategies. Through formulas (7.2), (7.3), (7.5) we find (with an accuracy to the third sign after comma):

$$p_1 = \\ = \frac{0,44-1}{0,608+0,44-1-1} = 0,588;$$
$$p_2 = 1 - p_1 = 0,412;$$
$$v = \frac{0,44 \cdot 0,608 - 1 \cdot 1}{0,608+0,44-1-1} = \\ = 0,768; \quad q_1 = 0,588; q_2 = 0,412.$$

(in this case $q_1 = p_1 = p_2$ on the strength of the fact that $a_{12} = a_{21}$.)

Thus, the optimal strategies of sides and worth of game are found:

$$S_A^* = (0,588,\ 0,412).$$
$$S_B^* = (0,588,\ 0,412),\ v = 0,768,$$

i.e. our optimal strategy lies in the fact that, into 58.8o/o of all cases (with probability of 0.588) making by carrier I, and into

41.2o/o of cases - II. Analogously enemy must with probability 0.588 attack the first bomber, and with probability 0.412 - the second. With this side A it will accomplish its task - to report bombs to target/purpose - with probability 0.768 that it is more lower worth of game 0.608 and lesser than upper pure value 1.

To the solution of game 2 x 2 it is possible to give convenient geometric interpretation. Let there be the game 2 x 2 with the matrix/die:

A_i \ B_j	B_1	B_2
A_1	a_{11}	a_{12}
A_2	a_{21}	a_{22}

Fig. 9.2.

Key: (1). Bombers. (2). Destroyer.

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Let us take the section of the axis of abscissas by the length of unit (Fig. 9.3). The left end of the section (point with abscissa x = 0) will represent strategy A_1, the right end of the section (x = 1) - strategy A_2; all the intermediate points of section will represent the mixed strategies of player A, moreover probability p_1 of strategy A_1 will be equal to distance from point S_A to the right end of the section, and probability p_2 of strategy A_2 - to a distance of left end. Let us lead through points A_1 and A_2 two perpendiculars to the axis of the abscissas: axis I-I and axis II-II. On axis I-I, let us plot/deposit gain with strategy A_1, while on axis II-II, - gains with strategy A_2.

Let the enemy apply strategy B_1; it gives on axes the I-I and II-II respectively point with ordinates a_{11} and a_{21}. Let us draw through these points the straight line B_1B_1. It is obvious, with any mixed strategy $S_A = (p_1, p_2)$ our gain will be expressed by point M on the straight line B_1B_1 conditionally let us call "strategy B_1".

It is obvious, accurately thus it can be constructed and strategy B_2 (Fig. 9.4).

We should find optimal strategy S_A^*, i.e., such, with which our minimum gain (with the worst for us behavior B) would be converted into maximum.

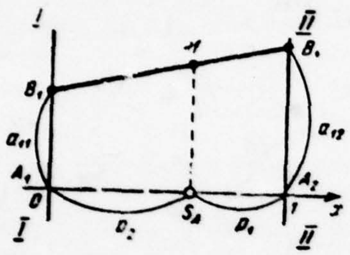

Fig. 9.3.

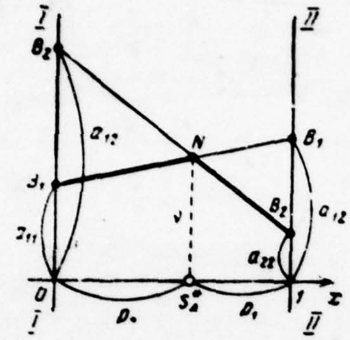

Fig. 9.4.

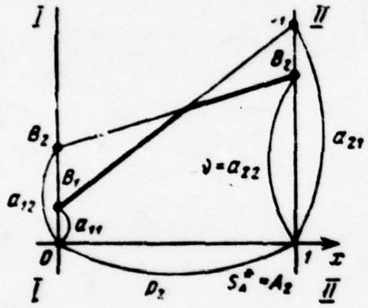

Fig. 9.5.

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For this, let us construct lower boundary of gain with strategies B_1, B_2, i.e., the broken line B_1NB_2, noted in Fig. 9.4 by heavy line. On this boundary will lie/rest player's minimum gain A with any of his mixed strategy; point N, and by which this gain it reaches maximum, and determines solution and worth of game. Not difficult to be convinced that ordinate of point N is nothing else but worth of game v, its abscissa is equal to p_2, but distance of the right end of the segment is equal p_1, i.e., distances from point S_A* to the ends of the segment are equal by probabilities p_2 and p_1 strategies A_2 and A_1 in the optimum mixed strategy of player A.

In our case the solution of game was determined by the point of intersection of strategies B_1, B_2; this will not always be thus. Figure 9.5 shows the case when the optimal strategy of player A is pure strategy A_2, although this does not correspond to the point of intersection of strategies. Here strategy A_2 of player A is clearly (with any strategy of the enemy) more suitable than strategy A_1. Figure 9.6 shows the case when the enemy has knowingly unsuitable strategy.

Geometric interpretation give the possibility to visually depict also the lower value of play α and upper β (Fig. 9.7). On this graph it is possible to give the geometric interpretation of optimum strategies of enemy B. Real/actually, it is not difficult to ascertain that portion/fraction q_1 of strategy B_1 in the optimum mixed strategy

$$S_B^* = (q_1, q_2)$$

is equal to the ratio of the length of segment KB_2 to the sum of the lengths of segments KB_2 and KB_1 on axis I-I:

$$q_1 = \frac{KB_2}{KB_2 + KB_1},$$

or, which is the same thing,

$$q_1 = \frac{LB_2}{LB_2 + LB_1}$$

on axis II-II.

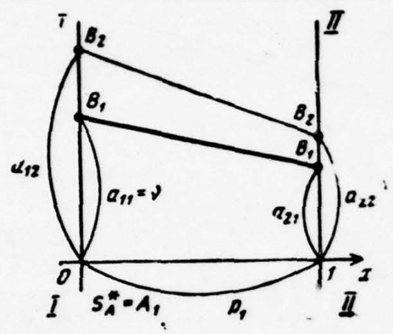

Fig. 9.6.

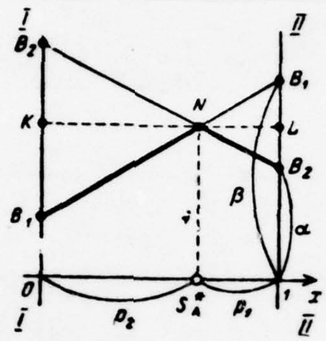

Fig. 9.7.

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Optimal strategy $S_B^*=(q_1, q_2)$ can be found and in another, direct manner, if to interchange the position of the players A and B, but instead of the maximum of lower boundary of gain to consider the minimum of upper bound (Fig. 9.8).

Figures 9.9 gives the geometric interpretation of solution of game "two bombers and a fighter" (example 2).

8. Games 2 x n and m x 2.

We ascertained that any game 2 x 2 can be solved by elementary methods. In perfect analogy can be solved any game 2 x n, where of us

(A) they have a total of two strategies, and of enemy (B) - an arbitrary number (n).

Thus, let there be the matrix/die of game 2 x n; it consist of two rows and n of columns. Analogous with the case 2 x 2 let us give to problem geometric interpretation; n of strategies of the enemy will be depicted n as straight lines (Fig. 9.10). Let us construct lower boundary of gain (broken line B_1MNB_2) and will find on it point N with maximum ordinate; this ordinate will be worth of game v, but the abscissa of point N will be equal to probability p_2 of strategy A_2 in the optimum mixed strategy of player A:

$$S_A^* = (p_1, p_2).$$

Knowing, which strategies intersect at point N, it is possible to indicate active strategies of enemy. In our case (Fig. 9.10) the optimum mixed strategy of the enemy

$$S_B^* = (0, q_2, 0, q_4)$$

consists of mixture of two active strategies B_2, B_4, intersecting of point N. Strategy B_3 is knowingly unfavorable, while strategy B_1 - unfavorable with optimum strategy S_A^*. Probabilities q_2 and q_4 are related as lengths of segments KB_4 and KB_2 in Fig. 9.10.

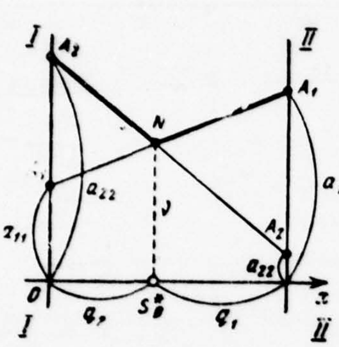

Fig. 9.8.

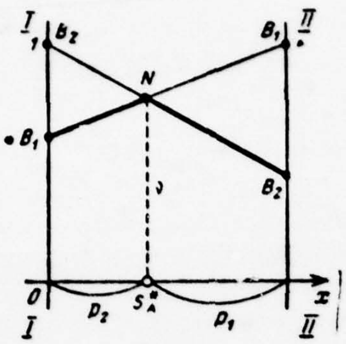

Fig. 9.9.

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If A will use its optimal strategy S_A^*, then gain will not be changed, whatever from its active strategies used B; however, it will be changed, if B will pass to strategies B_1 or B_3.

It is possible to demonstrate that of any final game m x n there is solution in which the number of active strategies of each side does not exceed smallest of the numbers m and n.

From this, in particular, it follows that at game 2 x n always has the solution, in which from each side participates not more than two active strategies. It is worth only finding these strategies - and game 2xn is converted into game 2 x 2 which is solved elementary.

Hence escape/ensues this practical method of the solution of game 2 x n: is constructed geometric interpretation (Fig. 9.10), is ought the pair of strategies, which intersect at point N (if in it intersects more than two strategies, is taken any pair) - these strategies represent by themselves player's active strategies B, and game 2 x n is reduced to game 2 x 2.

it is obvious, so can be solved game m x 2, with that difference, that it is constructed not the lower, and upper boundary of gain, and on it it is ought not the maximum, but the minimum (Fig. 9.11).

Example 1. Game "aircraft and the anti-aircraft guns".

Side A (aircraft) will attack for the objective, side B (the anti-aircraft guns) defends it. Of side A two aircraft, of side B - three antiaircraft weapons. Each aircraft is the carrier of the powerful damaging means: for the damage of object, it is sufficient so that to it would burst open at least one aircraft. Aircraft can choose for an approach to object any of three directions: I, II or III, without varying it subsequently (Fig. 9.12). Enemy (B) can place any of his instruments in any direction; each of the instruments

shoots through only the range of space, which relates to this direction, and do not shoot through adjacent directions. Each weapons can fire only one aircraft; the fired aircraft is surprised with full/total/complete authenticity. Side A does not know, where are placed instruments; side B does nct kncw, whence will arrive flying aircraft. Problem of side A - to strike object, sides B - not to allow damage/defeat. To find the solution of game.

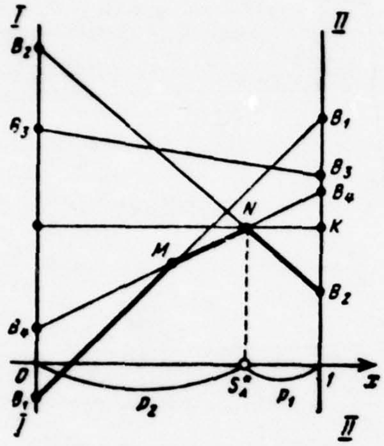

Fig. 9.10.

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Solution. If we as strategies examine all the possible methods of the selection of directions by aircraft and the arrangements of instruments, a quantity of strategies will be very greatly - 9 on the one hand even 27 on the other hand. However, it is possible to be bounded to the much smaller number of strategies, if previously they "are mixed" and are considered for A only two strategies:

A_1 - to send by one aircraft on two different (any) directions;

A_2 - to send both aircraft by one (to any) direction, and for an enemy - three strategies;

B_1 (1 + 1 + 1) - to place on one instrument on each direction;

B_2 (2 + 1 + 0) - to place two instruments on one (any) direction, one - on another, and the third to leave those not protected;

B_3 (3 + 0 + 0) - to place all three weapons on one (any) direction, and two others to leave unprotected.

In this case, it is assumed that the selection of each of the directions is produced randomly and with identical probability.

Let us comprise the matrix/die of game. Gain A in this case - kill probability of object, otherwise - probability that to object will burst open at least one aircraft.

Let us consider gains for all combinations of strategies.

1. A_1B_1 - aircraft they fly in different directions, instruments are arranged on one (1 + 1 + 1). Gain a_{11} - probability that at least one aircraft will burst open to object - in this case it is equal to zero: $a_{11} = 0$.

2. A_2B_1 - aircraft they fly in one and the same direction, instruments are arranged on one (1 + 1 + 1). It is obvious, with this one of the aircraft, being fired, for sure will burst open to the object: $a_{21} = 1$.

3. A_1B_2 - aircraft they fly on one; enemy places two instruments on one direction, one - to another is leaved not protected the third (2 + 1 + 0). In order to burst open to object, at least one of the aircraft it must select the unprotected direction. The probability of this event let us find through the probability of the opposite event: "both aircraft will select the defended direction".

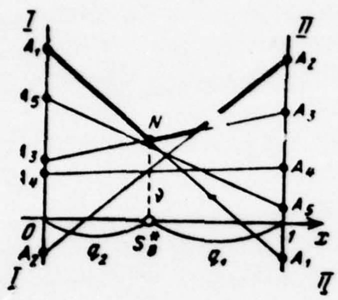

Fig. 9.11.

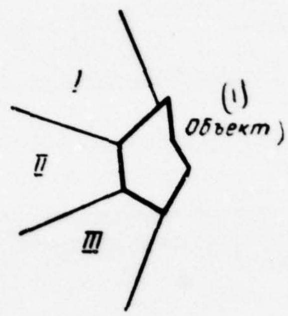

Fig. 9.12.

Key: (1). object.

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The probability of this event is equal to $2/3 \cdot 1/2 = 1/3$, whence probability of the interesting to us event: $a_{12} = I = 1/3 = 2/3$.

4. A_2B_2 - aircraft they fly together; instruments are placed according to pattern (1 + 2 + 0). Let us again find probability that both aircraft will be struck. For this, they must select the direction, defended by two instruments; probability this 1/3, the probability of the opposite event: $a_{22} = 1 - 1/3 = 2/3$.

5. A_1B_3 - aircraft they fly separately, instruments are placed

all three to one direction (3 + 0 + 0). It is obvious, in this case both aircraft are biased/beaten they cannot be, and a_{13} = 1.

6. A_2B_3 - aircraft fly together, instruments are placed all three to one direction (3 + 0 + 0). So that both the aircraft would be struck, they they must select the direction in which stand all three instruments. Probability this 1/3. Probability that at least one aircraft will burst open to object, will be a_{23} = 2/3.

We comprise the matrix/die of the game:

A_i \ B_j	B_1	B_2	B_3	α_i
A_1	0	2/3	1	0
A_2	1	2/3	2/3	2/3
β_j	1	2/3	1	

From matrix/die it is evident that the lower worth of game is equal to upper: $\alpha = \beta = v = 2/3$; that means game has saddle point and is solved in the pure strategies: side A (aircraft) must always use strategy A_2 (to fly together), and side B must always arrange instruments according to pattern (1 + 2 + 0), i.e., to place two weapons on some one direction, one instrument - in another, and one direction to leave with generally not protected.

Figures 9.13 gives the geometric interpretation of game.

Example 2 (version of the same game). Conditions are the same, but for side A are possible not three, but four directions of approach to object, but side B are disposed of four by instruments.

Solution. Of us as before two possible strategies:

A_1 - to send aircraft separately,

A_2 - to send aircraft together.

Of enemy five possible strategies:

B_1 - (1 + 1 + 1 + 1) - to place on one instrument on each direction,

B_2 - (2 + 1 + 1 + 0) - to place two instruments on one direction, on one - on two others and one to leave that not protected,

B_3 - (2 + 2 + 0 + 0) - to place on two instruments on two directions, and two to leave not protected,

B_4 (3 + 1 + 0 + 0) - to place three instruments on one direction, one - on another, and two to leave those not protected;

B_5 (4 + 0 + 0 + 0) - to place all four instruments on one direction, and remaining three to leave those nct protected.

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Strategies B_4 and B_5 can be previously reject/thrown as knowingly unfavorable. It is real/actual, if in one direction fly not more than two aircraft and each of them is surprised with probability one by one instrument - to place on one the direction of more than two instruments excessively. Discussing, as in previous example, let us construct the matrix/die of game.

A_i \ B_j	B_1	B_2	B_3	α_i
A_1	0	1/2	5/6	**1/2**
A_2	1	3/4	1/2	**1/2**
β_j	1	3/4	5/6	

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This game 2 x 3 does not have saddle point ($\alpha = 1/2$, $\beta = 3/4$). We seek solution in mixed strategies. Is isolated active strategies of the enemy: this B_1 and B_3 (Fig. 9.14). After this game it is reduced to game 2 x 2:

A_i \ B_j	B_1	B_3
A_1	0	5/6
A_2	1	1/2

Solving this game, we find the optimal strategies of the sides:

$$p_1 = {}^3/_8;\quad p_2 = {}^5/_8;\quad v = {}^5/_8;\quad q_1 = {}^1/_4,\quad q_3 = {}^3/_4;$$
$$S_A{}^* = ({}^3/_8,\ {}^5/_8);\quad S_B{}^* = ({}^1/_4,\ 0,\ {}^3/_4).$$

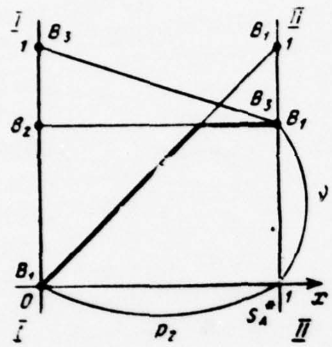

Fig. 9.13.

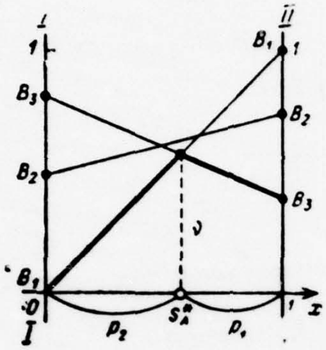

Fig. 9.14.

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Thus, it is possible to formulate following recommendations to sides A and B: side A must with probability 3/8 send aircraft separately, and with probability 5/8 - together; side B must with probability 1/4 apply the arrangement of instruments (1 + 1 + 1 + 1), and with probability 3/4 - arrangement (2 + 2 + 0 + 0). In this case, the gain (kill probability of object) is equal to $v = 5/8$, that it is more than lower worth of game and lesser than upper.

Example 3. Game is the "distribution of forces in onset and defense".

To side A, which disposes of three battalions of infantry, it

attempts to take certain object B; side B, which disposes of four battalions of infantry, attempts to prevent this. Each of the attacking/advancing battalions can be directed toward object along any of two roads: I and II (Fig. 9.15). Side B also can place any of its battalions to any of the roads. If on the road the forces of side B meet superior forces of side A, the latter push aside defense, they pass to object and occupy it; if on road defense outnumbers attack, attack is repulsed, the forces of side A will move away and no longer they renew attack. If on road are encountered the forces of identical number, attack is repulsed, the forces of side A will move away and no longer they renew attack. If on road are encountered forces of identical number, side A with probability 0.4 conquers and passes to object, and with probability 0.6 attack turns out to be repeled.

It is required to give recommendation for sides regarding the quantity of battalions which should be directed toward each of the roads.

Solution. Gain A in this case - probability of the occupation of object. Let us consider following strategies of attack (A):

$A_1(2 + 1)$ - to direct two battalions along one of the roads (any) and one - on another;

$A_2(3 + 0)$ - to direct all three battalions along one of the roads (any).

Strategies of defense (B) will be:

$B_1(3 + 2)$ - to direct along two battalion toward each of the roads;

$B_2(3 + 1)$ - to direct three battalions toward one of the roads (any), and one - toward another;

B_3 $(4 + 0)$ - to direct all four battalions toward one of the roads (any, and another road to leave not protected.

Let us comprise the matrix/die of game. let us find gain for all combinations of strategies.

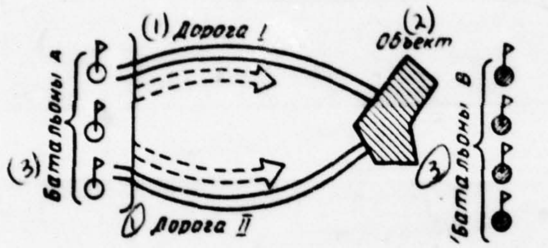

Fig. 9.15.

Key: (1). Road. (2). Object. (3). Battalions.

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1. A_1B_1. On one road they meet one battalion of attack two battalions of defense; attack on this road is repulsed. On another road meet two battalions of attack two defenses; according to condition the attack conquers with probability 0.4: $a_{11} = 0.4$.

2. A_2B_1. In this case, on one of the roads with full/total/complete authenticity, there will be the preponderance of the forces of attack, and $a_{21} = 1$.

3. A_1B_2. Since the selection of any road for each side is equally probable, then with probability 1/2, on one road will be met

two battalions A three B, to another - one battalion A one B; on the first road the attack will be repeled, to another - will occur the occupation of object with probability 0.4. With the same probability 1/2 will be met on one road one battalion A three B, to another - two battalions A one B, and object will be occupied with full/total/complete authenticity. Applying the formula of composite probability, we find:

$$a_{12} = 1/2 \cdot 0,4 + 1/2 \cdot 1 = 0,7.$$

4. A_2B_2. With probability 1/2 on one road, will be met three battalions A three B, to another - collision will not be; in this case the probability of the occupation of object 0.4. With the same probability 1/2 three battalions A will be met only one battalion B, they pass and will be occupied object. On the formula of the composite probability:

$$a_{22} = 1/2 \cdot 0,4 + 1/2 \cdot 1 = 0,7.$$

5. A_1B_3. Since forces A go along two roads, and forces B are arrange/located only on one of the roads, side A with authenticity will occupy the object: $a_{13} = 1$.

6. A_2B_3. With the probability of 1/2 forces A, they will go along that road where there is no defense, and will be occupied object; with probability 1/2, they will be repeled by superior forces

of defense; hence

$$a_{23} = 1/2 \cdot 1 + 1/2 \cdot 0 = 0{,}5.$$

The matrix/die of game 2 x 3 takes the form:

A_i \ B_j	B_1 (2+2)	B_2 (3+1)	B_3 (4+0)
A_1 (2+1)	0,4	0,7	1
A_2 (3+0)	1	0,7	0,5

Lower worth of game α = 0.5, upper pure value β = 0.7; game does not have saddle point. We seek solution in mixed strategies. The gecmetric interpretation of game is given in Fig. 9.16. Lower boundary of gain reaches maximum in pcints N' and N" on all section between them; this maximum there is worth of game $v = 0{,}7$. In this case the solution of game was obtained ambiguous: side A can use any of its mixed strategies, which correspond to the pcints of the axis of abscissas from K' to K".

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Thus, of side A there is a nondenumerable set of the optimal strategies. Let us find the abscissas of pcints N' and N". They will be equal to respectively p_2' and p_2" - to the boundaries in which is included the probability of strategy A_2 in the optimum mixed strategy

of player A. From drawing we have:

$$\frac{0,7-0,4}{p_2'}=\frac{1-0,7}{1-p_2'},$$

whence $p_2' = 0.5$. Analogously we obtain

$$\frac{1-0,7}{p_2''}=\frac{0,7-0,5}{1-p_2''},$$

whence $p_2'' = 0.6$.

Thus, as the optimum mixed strategy side A can apply any $S_A^*(p_1, p_2)$, at which probabilities p_1 and p_2 lie/rest: the first - between 0.4 and 0.5; the second respectively between 0.6 and 0.5.

It goes without saying that extreme values p_1 and p_2 also give the optimal strategies of player A:

$$S_A^* = (0,5,\ 0,5),\quad S_B^* = (0,4,\ 0,6).$$

Thus, the optimal strategy of player A is found: it lies in the fact that, with the probability, which takes any value between 0.4 and 0.5, directing two battalions on one of the roads (any), but the of remaining battalions - along another road; in all remaining cases to send all three battalions by one of the roads (any).

Concerning the optimal strategy of enemy (B), then, as can be seen from Fig. 9.16, it is reduced to the application/use of only one

pure strategy, namely B_2:

$$S_B^* = (0, 1, 0),$$

i.e. defending always must display three battalions on one road (any), and one battalicn - cn another rcad. Worth of game, i.e., the stable gain of side A in this case will be equal to upper pure value C.7 [1]).

FCCTNOTE [1]. From Fig. 9.16 it is pcssible to dc the conclusion that active strategies of side B, besides B_{12}, they are even B_1 and B_3 (since the corresponding lines of strategies intersect at points N' and N"). However, it is not difficult tc ascertain that since line B_2B_2 is parallel to the axis of abscissas, the actual pcrtion/fraction of strategies B_1 and B_3 in the optimum mixed strategy of the defense should be equal to zero. ENDFOOTNOTE.

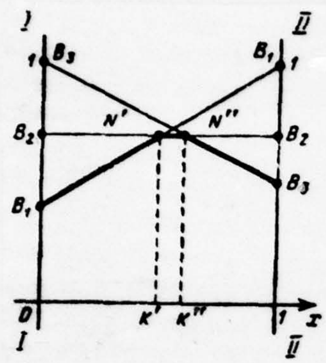

Fig. 9.16.

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9. Solution of games m x n.

Until now, we examined only elementary games 2 x n and m x 2, for which there is a simple geometric interpretation, which makes it possible to solve these games with the help of the simplest methods.

In the case of games 3 x n (or m x 3) a similar geometric interpretation also can be constructed, but instead of the plane it becomes three-dimensional/space and much less demonstrative. In the case of game m x n, where m > 3, n > 3, from geometric interpretation it is necessary to refuse and into force enter purely the calculated methods of the solution of games.

In the general case, with large m and n, the solution of game m x n represents a sufficiently laborious problem, but fundamental difficulties it does not contain. It is easy to show that the solution of any of the final game m x n can be reduced to the already known to us problem of linear programming (main 2).

It is real/actual, let us consider game m x n with player's m strategies $A_1, A_2, \ldots, A_m$ A and n by player's strategies $B_1, B_2, \ldots, B_n$ B. Is assigned the matrix/die of game (a_{ij}).

Is required to find the solution of game, i.e., two optimum mixed strategies of the players A and B:

$$S_A^* = (p_1, p_2, \ldots, p_m); \quad S_B^* = (q_1, q_2, \ldots, q_n),$$

where

$$p_1, p_2, \ldots, p_m; q_1, q_2, \ldots, q_n \quad (p_1 + p_2 + \ldots + p_m = 1; \ q_1 + q_2 + \ldots + q_n = 1)$$

- the probability of application of the pure strategies (some of them, that correspond to inactive strategies, can be equal to zero).

Let us find first optimal strategy S_A^*. This strategy must ensure to us the gain, not less v, with any behavior of the enemy,

and the gain, equal to v, with his optimum behavior (i.e. with strategy S_B*).

Worth of game v to us thus far is unknown. without breaking the generality, it is possible to assume to its equal certain positive number $v>0$. It is real/actual, so that would be implemented condition $v>0$, it is sufficient so that all matrix elements (a_{ij}) would be nonnegative. This always it is possible to attain, adjoining KO to all matrix elements (a_{ij}) one and the same sufficiently high positive value M; with this worth of game it will be increased by M, but solution it will not be changed. Thus, let us count $v>0$.

Let us assume that we (A) apply our optimal strategy S_A*, and enemy (B) our pure strategy B_j. Then our average gain is equal to:

$$a_j = p_1 a_{1j} + p_2 a_{2j} + \dots + p_m a_{mj} \ (j=1, \dots, n).$$

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Our optimal strategy S_A* possesses that property, which with any behavior of enemy ensures to us the gain, not less than worth of game v; that means that any of numbers a_j cannot be less v. We obtain a series of the conditions:

$$\left.\begin{array}{l} p_1 a_{11} + p_2 a_{21} + \dots + p_m a_{m1} \geqslant v, \\ p_1 a_{12} + p_2 a_{22} + \dots + p_m a_{m2} \geqslant v, \\ \dots\dots\dots\dots\dots \\ p_1 a_{1n} + p_2 a_{2n} + \dots + p_m a_{mn} \geqslant v. \end{array}\right\} \quad (9.1)$$

Let us divide inequality (9.1) into positive value ν and let us introduce the designations:

$$x_1 = \frac{p_1}{\nu}, \quad x_2 = \frac{p_2}{\nu}, \ \ldots, \ x_n = \frac{p_n}{\nu}. \tag{9.2}$$

Then conditions (9.1) are registered in the form:

$$\left.\begin{array}{l} a_{11} x_1 + a_{21} x_2 + \ \ldots \ + a_{m1} x_m \geqslant 1, \\ a_{12} x_1 + a_{22} x_2 + \ \ldots \ + a_{m2} x_m \geqslant 1, \\ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \\ a_{1n} x_1 + a_{2n} x_2 + \ \ldots \ + a_{mn} x_m \geqslant 1, \end{array}\right\} \tag{9.3}$$

where $x_1, x_2, \ldots, x_m$ - nonnegative variables. In force (9.2) and the fact that $p_1 + p_2 + \ldots + p_m = 1$, the variables $x_1, x_2, \ldots, x_m$ satisfy the condition

$$x_1 + x_2 + \ \ldots \ + x_m = \frac{1}{\nu}. \tag{9.4}$$

We wish to do our guaranteed gain by maximally possible; it is obvious, in this case, right side (9.4) takes minimum value. Thus, the problem of the solution of game was reduced to following mathematical task.

To determine the nonnegative values of variables $x_1, x_2, \ldots, x_m$ so that they would satisfy linear limitations (9.3) and in this case their linear function

$$L = x_1 + x_2 + \ \ldots \ + x_m \tag{9.5}$$

was converted into the minimum.

Before us - the typical problem of linear programming.

Thus, solving the problem of linear programming, we can find optimal strategy S_A* of player A.

Let us find now optimal strategy S_B* of player B. Everything will be analogous with the solution of game for player A, with that difference, that player B attempts not maximize, but minimize gain, but that means that not to minimize, but to maximize value $1/v$. Instead of conditions (9.3) it had to be observed the condition:

$$\left.\begin{array}{l} a_{11}y_1+a_{12}y_2+\ \dots\ +a_{1n}y_n\leqslant 1, \\ a_{21}y_1+a_{22}y_2+\ \dots\ +a_{2n}y_n\leqslant 1, \\ \dots\dots\dots\dots\dots \\ a_{n1}y_1+a_{n2}y_2+\ \dots\ +a_{nn}y_n\leqslant 1, \end{array}\right\} \qquad (9.6)$$

where $y_1, y_2, \dots, y_n$ - the nonnegative variables, equal to

$$y_1=q_1/v,\quad y_2=q_2/v,\quad \dots,\quad y_n=q_n/v$$

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It is required so to select variables $y_1, y_2, \dots, y_n$, so that they would satisfy conditions (9.6) and would convert into maximum the linear function

$$L=y_1+y_2+\ \dots\ +y_n=\frac{1}{v} \qquad (9.7)$$

Instead of maximizing of function (9.7), it is possible, as we know that to minimize the function

$$L'=-L=-y_1-y_2-\ \dots\ -y_n=-\frac{1}{v}. \qquad (9.8)$$

Thus, we reduced the problem of the solution of any of the final game m x n to the pair of the problems cf linear programming; the methods of the solution of such problems to us are already well known (see Chapter 2).

Let us incidentally note that from the information of the problem of the solution of game to the problem of linear programming escape/ensue the considerations apropos of the existence of the solution of game m x n.

It is real/actual, let the problem of the determination of optimal strategy $S_{A}^{*} = (p_1, p_2, \ldots, p_m)$ of player A be reduced to the problem of linear programming with condition-inequalities (9.3) and minimized function (9.5). Always whether there is its solution? We know (see Chapter 2) that the solution of the problem of linear programming can and not to exist; it it is absent, if:

1) conditions (equality or inequality) not at all have the permissible nonnegative solutions;

2) the permissible solutions exist, but among them no optimum,

since the minimized function is not limited from below.

Let us look how is matter in our case. It is not difficult to ascertain that the permissible solution of ZLP in our case always exists. It is real/actual, let us do matrix elements (a_{ij}) by the strictly positive (for this is sufficient to adjoin to each of them sufficiently large Mach number) and let us designate the smallest matrix element (a_{ij}) through μ:

$$\mu = \min_{i} \min_{j} a_{ij}.$$

is placed now $x_1 = 1/\mu,\ x_2 = x_3 = \ldots = x_m = 0.$ It is not difficult to see that this system of the values of variables $x_1, x_2, \ldots, x_m$ represents by themselves the permissible solution of ZLP - they all are nonnegative, and their set it satisfies conditions (9.3).

Let us now ascertain that linear function (9.5) cannot be limited from below. It is real/actual, all $x_1, x_2, \ldots, x_m$ are nonnegative, and the coefficients of them in expression (9.5) are positive, which means, that function L in formula (9.5) is also nonnegative, which means, that it is limited from below (zero) and the solution of the problem of linear programming (and, consequently, of game m x n) exists.

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Example 1. to find by the method of linear programming the solution of game "three finger/pins" from example 2 of § 4.

Solution. The matrix/die of game takes the form:

A_i \ B_j	B_1	B_2	B_3
A_1	2	−3	4
A_2	−3	4	−5
A_3	4	−5	6

(9.9)

Adding to all matrix elements one and the same Mach number = 0.5, let us do them nonnegative:

A_i \ B_j	B_1	B_2	B_3
A_1	7	2	9
A_2	2	9	0
A_3	9	0	11

(9.10)

With this worth of game it will be increased by 5, but solution will not be changed. Let us designate new worth of game $v' = v + 5$. Let us find optimum mixed strategy $S_A^* = (p_1, p_2, p_3)$ of player A. Conditions (9.3) take the form:

$$\left.\begin{aligned} 7x_1 + 2x_2 + 9x_3 &> 1, \\ 2x_1 + 9x_2 &> 1, \\ 9x_1 + 11x_3 &> 1. \end{aligned}\right\} \qquad (9.11)$$

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Is minimumized linear function

$$L = x_1 + x_2 + x_3. \qquad (9.12)$$

Let us pass from condition-inequalities (9.11) to condition-equalities:

$$\begin{aligned} y_1 &= -1 - (-7x_1 - 2x_2 - 9x_3), \\ y_2 &= -1 - (-2x_1 - 9x_3), \\ y_3 &= -1 - (-9x_1 - 11x_3). \end{aligned}$$

Let us fill a simplex-table (Table 9.1). Since absolute terms are negative, then, set/assuming $x_1 = x_2 = x_3 = 0$, we will not obtain supporting/reference solution.

Table 9.1.

	Absolute term	x_1	x_2	x_3
y_1	-1	-7	-2	-9
y_2	-1	-2	-9	0
y_3	-1	-9	0	-11

Table 9.2.

	Absolute term	x_1	x_2	x_3 $y_3 \leftrightarrow$
y_1	-1 $\frac{9}{11}$	-7 $\frac{81}{11}$	-2 0	-9 $-\frac{9}{11}$
y_2	-1 0	-2 0	-9 0	0 0
$x_3 \leftrightarrow$ y_3	-1 $\frac{1}{11}$	-9 $\frac{9}{11}$	0 0	-11 $-\frac{1}{11}$

Table 9.3.

	Свободный член (1)	x_1	x_2 ($y_2 \leftarrow$)	y_3
y_1	$-\frac{2}{11}$ $\frac{2}{9}$	$\frac{4}{11}$ $\frac{4}{9}$	-2 $-\frac{2}{9}$	$-\frac{9}{11}$ 0
$x_2 \leftarrow$ y_2	-1 $\frac{1}{9}$	-2 $\frac{2}{9}$	-9 $-\frac{1}{9}$	0 0
x_3	$\frac{1}{11}$ 0	$\frac{9}{11}$ 0	0 0	$-\frac{1}{11}$ 0

Key: (1). Absolute term.

Table 9.4.

	Свободный член (1)	x_1	y_2	y_3
y_1	$\frac{4}{99}$	$\frac{80}{99}$	$-\frac{2}{9}$	$-\frac{9}{11}$
x_2	$\frac{1}{9}$	$\frac{2}{9}$	$-\frac{1}{9}$	0
x_3	$\frac{1}{11}$	$\frac{9}{11}$	0	$-\frac{1}{11}$

Key: (1). Absolute term.

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Applying the apparatus of the simplex method (see Chapter 2, § 7), we find supporting/reference solution (Tables 9.2, 9.3, 9.4). Table 9.4 gives the supporting/reference solution of OZLP:

$$x_1 = y_2 = y_3 = 0;\quad y_1 = {}^4/_{99},\quad x_2 = {}^1/_9,\quad x_3 = {}^1/_{11}.$$

It is necessary to check that is it optimum, i.e., is converted into the minimum expression (9.12). Using Table 9.4, is expressed L through the new independent variables x_1, y_2, y_3:

$$L = x_1 + x_2 + x_3 = x_1 + (1/9 - 2/9\,x_1 + 1/9\,y_2) + (1/11 - 9/11\,x_1 + 1/11\,y_3) =$$
$$= 20/99 - (4/99\,x_1 - 1/9\,y_2 - 1/11\,y_3)$$

let us register absolute term and the ccefficients of x_1, y_2, y_3 in upper row Table 9.5. From the fact that the coefficient of x_1 is positive it is apparent that increase x_1 reduces L, i.e., the minimum is not still achieve/reached. We produce replacement $x_1 \leftrightarrow y_1$ (Tables 9.5, 9.6).

Table 9.5.

$y_1 \leftrightarrow$

	Свободный член (1)	x_1	y_2	y_3
L	$\frac{20}{99}$ $\frac{-1}{5\cdot 99}$	$\frac{4}{99}$ $\boxed{-\frac{1}{20}}$	$-\frac{1}{9}$ $\frac{1}{90}$	$-\frac{1}{11}$ $\frac{9}{220}$
$x_1 \leftrightarrow$ y_1	$\boxed{\frac{4}{99}}$ $\frac{1}{20}$	$\frac{80}{99}$ $\frac{99}{80}$	$\boxed{-\frac{2}{9}}$ $-\frac{11}{40}$	$\boxed{-\frac{9}{11}}$ $-\frac{81}{80}$
x_2	$\frac{1}{9}$ $-\frac{1}{90}$	$\frac{2}{9}$ $\boxed{-\frac{11}{40}}$	$-\frac{1}{9}$ $\frac{11}{180}$	0 $\frac{9}{40}$
x_3	$\frac{1}{11}$ $-\frac{9}{220}$	$\frac{9}{11}$ $\boxed{-\frac{81}{80}}$	0 $\frac{9}{40}$	$-\frac{1}{11}$ $\frac{729}{880}$

Key: (1). Absolute term.

Table 9.6.

	Свободный член (1)	y_1	y_2	y_3
L	$\frac{1}{5}$	$-\frac{1}{20}$	$-\frac{1}{10}$	$-\frac{1}{20}$
x_1	$\frac{1}{20}$	$\frac{99}{80}$	$-\frac{11}{40}$	$-\frac{81}{80}$
x_2	$\frac{1}{10}$	$-\frac{11}{40}$	$-\frac{1}{20}$	$\frac{9}{40}$
x_3	$\frac{1}{20}$	$-\frac{81}{80}$	$\frac{9}{40}$	$\frac{59}{80}$

Key: (1). Absolute term.

Table 9.6 shows that function L takes the minimum value L_{min} = 1/5 with

$$y_1 = y_2 = y_3 = 0,$$
$$x_1 = {}^1/_{20}, \quad x_2 = {}^1/_{10}, \quad x_3 = {}^1/_{20}.$$

Hence $v' = 1/L_{min}$ = 5, i.e., worth of game with matrix/die (9.10) $v' = 5$.

Consequently, the value of initial game with matrix/die (9.9):

$$v = v' - 5 = 0.$$

This value of gain is reached at

$$x_1 = {}^1/_{20}, \quad x_2 = {}^1/_{10}, \quad x_3 = {}^1/_{20},$$

i.e. for the probabilities of strategies

$$p_1 = x_1 v' = {}^1/_4; \quad p_2 = x_2 v' = {}^1/_2; \quad p_3 = x_3 v' = {}^1/_4.$$

Thus, is found the solution of game - optimal strategy of player A:

$$S_A^* = ({}^1/_4, {}^1/_2, {}^1/_4)$$

and worth of game v = 0.

The optimal strategy of player B can be found accurately thus, if we comprise the conditions, analogous (9.7), but not for chairs,

but for rows, after replacing in them signs $\geqslant$ by $\leqslant$, and value L to convert not into the minimum, but into maximum. However, in this case for this, the necessities no: from the symmetry of rows and columns of matrix/die it is clear that the optimal strategy of player B must be the same as the optimal strategy of player A:

$$S_B^* = (1/4,\ 1/2,\ 1/4).$$

Thus, in game "three finger/pins" the optimal strategy of each of the players lies in the fact that, with probability of 1/4 showing of one finger/pin, with probability 1/2 - two finger/pins and with probability 1/4 - three finger/pins. In this case, each player's average gain will be equal to zero $(v = 0)$

In this example each player's all three strategies were active. Such game, in which all strategies are active, is called completely averaged. In a following example we will consider an example of the completely not averaged game.

Example 2. Game is "armament - interference".

Side A disposes of three forms of armament A_1, A_2, A_3, while side B - by three forms of interferences B_1, B_2, B_3. The probability of the solution of combat mission by side A in the various forms of armament and interferences is assigned by the matrix/die:

B_j / A_i	B_1	B_2	B_3
A_1	0,8	0,2	0,4
A_2	0,4	0,5	0,6
A_3	0,1	0,7	0,3

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Side A attempts to solve combat mission, side B - to prevent this to find the optimal strategies of sides.

Solution. Getting rid from fractions, let us rewrite matrix/die in the form:

B_j / A_i	B_1	B_2	B_3
A_1	8	2	4
A_2	4	5	6
A	1	7	3

let us designate the value of new game with this matrix/die $v' = 10v$. Let us register conditions (9.3):

$$\begin{aligned} 8x_1 + 4x_2 + x_3 &\geq 1, \\ 2x_1 + 5x_2 + 7x_3 &\geq 1, \\ 4x_1 + 6x_2 + 3x_3 &\geq 1, \end{aligned}$$

whence (passing to condition-equalities)

$$\left.\begin{aligned} y_1 &= -1 - (-8x_1 - 4x_2 - x_3), \\ y_2 &= -1 - (-2x_1 - 5x_2 - 7x_3), \\ y_3 &= -1 - (-4x_1 - 6x_2 - 3x_3). \end{aligned}\right\} \quad (9.13)$$

It is required to find the nonnegative values x_1, x_2, x_3, y_1, y_2, y_3, which satisfy conditions (9.13) and which rotate in the minimum the linear function:

$$L = 1/v = x_1 + x_2 + x_3.$$

We solve problem by the simplex method (by lowering details, let us give immediately optimum solution, Table 9.7).

Table 9.7 shows that minimum L is achieve/reached and equal to L_{min} = 7/32. This value is reached at

$$y_1 = x_3 = y_2 = 0; \quad x_1 = {}^1/_{32}, \quad y_3 = {}^1/_4, \quad x_2 = {}^3/_{16}.$$

Table 9.7.

	(1) Свободный член	y_1	x_3	y_2
L	$\frac{7}{32}$	$-\frac{3}{32}$	$-\frac{1}{32}$	$-\frac{1}{8}$
x_1	$\frac{1}{32}$	$-\frac{5}{32}$	$-\frac{23}{32}$	$\frac{1}{8}$
y_3	$\frac{1}{4}$	$-\frac{1}{4}$	$\frac{17}{4}$	-1
x_2	$\frac{3}{16}$	$\frac{1}{16}$	$\frac{27}{16}$	$-\frac{1}{4}$

Key: (1). Absolute term.

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We hence find probabilities p_1, p_2, p_3, with which player A must apply his strategies A_1, A_2, A_3:

$$p_1 = x_1 v', \quad p_2 = x_2 v', \quad p_3 = x_3 v'$$

and worth of game:

$$v' = 1/L_{min}.$$

In this case,

$$v' = {}^{32}/_7; \quad p_1 = {}^1/_{32} \cdot {}^{32}/_7 = {}^1/_7;$$
$$p_2 = {}^3/_{16} \cdot {}^{32}/_7 = {}^6/_7; \quad p_3 = 0 \cdot {}^{32}/_7 = 0.$$

Thus, the optimal strategy of player A is found:

$$S_A^* = ({}^1/_7, {}^6/_7, 0).$$

i.e., we must use with probability 1/7 by first form of armament, with probability 6/7 - by the second, but the third form of armament not to apply at all. In this case, the probability of the execution of combat mission will be maximum:

$$v = v'/_{10} = {}^{32}/_{70} \approx 0,457.$$

Let us now find the optimal strategy S_B^* of enemy. In the general case for this, it is necessary to enter in the manner that it is said above: to solve problem for an enemy in the manner that we it solved for ourselves, with the replacement of the columns of matrix/die by rows, signs $\geqslant$ on $\leqslant$ and the minimum by maximum. However, in this case for this, the necessities no: us helps the fact that to us are already known player's active strategies A, and their only two: A_1 and A_2. Game, thus, became game 2 x 3, which can be solved elementary. Lowering details, let us give only the solution:

$$S_B^* = (^3/_7, \, ^4/_7, \, 0).$$

i.e. the optimal strategy of enemy lies in the fact that, with probability of 3/7 using interferences B_1, with probability 4/7, - by interferences B_2, but the third form of interferences (B_3) not to apply at all.

In conclusion let us note that the demonstrated in this paragraph general method of the solution of games m x n (information to the problem of linear programming) does not always turn out to be the simplest. Frequently game - especially with small m and n - can

be solved simpler, if we previously guess, which strategies are active. For example, if the matrix/die of game - is square ($m = n$), then it is possible to try - it is not game completely averaged? In this case, all strategies of both sides are active, and inequalities (9.3) are converted into equalities. If, after solving this system of equations, we will obtain the positive values $x_1, x_2, \ldots, x_m$, then, by store/adding up them, let us find value $1/v$:

$$x_1 + x_2 + \ldots + x_m = 1/v,$$

whence worth of game:

$$v = \frac{1}{x_1 + x_2 + \ldots + x_m},$$

and probabilities $p_1, p_2, \ldots, p_m$ in the optimal strategy S_A^* will be located as

$$p_1 = x_1 v, \quad p_2 = x_2 v, \quad \ldots, \quad p_m = x_m v.$$

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10. Solution of final games by the method of iterations.

In practical problems frequently there is no need for finding the exact solution of game; sufficiently there is to find approximate solution, which ensures the average gain, close to worth of game.

Tentatively worth of game v can be determined directly from matrix/die, knowing lower worth of game α and upper β. If α and β are close, then virtually there is no necessity to be occupied by the searches of exact solution, and it suffices it will be as optimum take pure/clean minimax strategies. In the same cases, when α and β are not close, approximate solution of game can be obtained, using the method of iterations.

The idea of this method is reduced to following. Is developed "thought experiment", in which both sides A and B apply against each other their strategies. Experiment consists of the sequence of the separate "party/batches" of this game. Begins it from the fact that one of the players (let us say A or "we") is chosen arbitrarily one of its strategies, for example A_i. Enemy (B) this answers that of his strategies B_j, that is least advantageous for us, i.e., is converted

gain with strategy A_i into the minimum. This course we answer that our strategy A_k which gives maximum gain with strategy of enemy B_j. Further - again the turn of enemy. It answers our pair of courses A_i and A_k that its strategy B_{i1} which gives the smallest average gain to one party/batch with these two strategies and, etc. At each step/pitch of iterative process, each player answers the next course of other that his strategy, which is optimum relative to all previous courses of enemy, considered as certain "mixed strategy", in which the pure strategies enter in the proportions, determined by the frequency of their application/use.

This method of the construction of the optimal strategies represents by itself certain model of the practical "mutual instruction" of the players when each of them on experiment "probes" the way of behavior of the enemy tries to answer it the best for itself form.

It is possible to demonstrate that the process of iterations descends; if this alternating sequence of party/batches to continue is sufficient for long, then the average gain, which is necessary to one party/batch, will approach the worth of game v, and of frequency $P_1^*, P_2^*, \ldots, P_m^*$; $Q_1^*, Q_2^*, \ldots, Q_n^*$ with which were applied strategies $A_1, A_2, \ldots, A_m$; $B_1, B_2, \ldots, B_n$ in this "drawing", they will approach probabilities $p_1, p_2, \ldots, p_m$ $q_1, q_2, \ldots, q_n$ in the

optimum mixed strategies: $S_A^* = (p_1, p_2, \ldots, p_m)$; $S_B^* = (q_1, q_2, \ldots, q_n)$.

Calculations show that the convergence of method - very slow; however, for high speed ETsVM [ЭЦВМ - digital computer] this is not serious obstruction. The advantage of the method of iterations lies in the fact that its complexity comparatively little grows with an increase in the size/dimension of table m x n, whereas the complexity of the solution of the problem of linear programming sharply increases with increase in m and n.

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Let us demonstrate the application/use of an iterative method based on the example of game "three finger/pins", solved by us accurately in the previous paragraph. The fact that we know the solution of game and its value (v = 5), will aid us to consider the accuracy of the method of iterations.

Example. To solve by the method of iterations game with the matrix/die

B_j / A_i	B_1	B_2	B_3
A_1	7	2	9
A_2	2	9	0
A_3	9	0	11

Solution. Table 10.1 gives the first of 30 step/pitches of the process of iterations. In the first column is given the number of party/batch (pair of selections) k, in the second - number i of selected in this party/batch strategy of player A. In the subsequent three columns - "accumulated gain" for first k of party/batches with those strategies which were applied player's both in the previous party/batches, with strategy A_i of player A in this party/batch and with strategies B_1, B_2, B_3 of player B in this party/batch. From these accumulated gains is emphasized minimum (if such minimum gains several, then are stressed they everything). The emphasized number determines by itself player's most advantageous strategy B in this party/batch - it corresponds to the number of that strategy B_j, for which is reached the minimum of the accumulated gain (if such minimums several, is taken any of them, for example, by random drawing). Thus is enter/written in following column the number of optimum reciprocal strategy of enemy j. In the subsequent three columns is given the accumulated gain for k of party/batches respectively with strategies A_1, A_2, A_3 of player A. From these values

is written maximum; it determines by itself the selection of player's strategy A in following party/batch (following table row). In further columns table 10.1 are placed these data:

$\underline{v}$ - minimum accumulated gain, divided into the number of party/batches k; $\bar{v}$ - maximum accumulated gain, divided into the number of party/batches k; $v^* = \frac{\underline{v}+\bar{v}}{2}$ - their arithmetic mean (is placed in table between $\underline{v}$ and $\bar{v}$).

Value v^* can serve (is better than $\underline{v}$ and $\bar{v}$) as approximate value of worth of game.

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Table 10.1

k	l	h	B_k	B	j	A_1	A_2	A_3	$\underline{v}$	v^*	$\bar{v}$
1	3	9	$\underline{\text{[illegible]}}$	11	2	2	$\overline{9}$	0	0	4,5	9
2	2	11	$\underline{9}$	11	2	4	$\overline{18}$	0	4,5	6,75	9
3	2	13	18	$\underline{11}$	3	13	$\overline{18}$	11	3,67	[illegible],84	6
4	2	15	27	$\underline{11}$	3	22	18	$\overline{22}$	2,75	4,13	5,50
5	1	22	29	$\underline{20}$	3	$\underline{31}$	18	$\overline{33}$	4,00	5,30	6,60
6	3	31	$\underline{29}$	31	2	33	27	$\overline{33}$	4,81	5,17	5,50
7	1	38	$\underline{31}$	40	2	35	$\overline{36}$	33	4,43	4,79	5,14
[illegible]	2	$\underline{40}$	$\underline{40}$	$\underline{40}$	2	$\underline{37}$	$\overline{45}$	33	5,00	5,30	5,61
9	2	42	49	$\underline{40}$	3	$\underline{46}$	45	44	4,45	4,78	5,11
10	1	$\underline{49}$	51	$\underline{49}$	1	53	47	$\overline{53}$	4,90	5,10	5,30
11	3	58	$\underline{51}$	60	2	55	$\overline{56}$	53	4,64	4,87	5,09
12	2	$\underline{60}$	$\underline{60}$	$\underline{60}$	2	$\underline{57}$	$\overline{65}$	53	5,00	5,20	5,41
13	2	62	69	$\underline{60}$	3	$\underline{66}$	65	64	4,61	4,84	5,07
14	1	$\underline{69}$	71	$\underline{69}$	1	73	67	$\overline{73}$	4,93	5,07	5,21
15	3	78	$\underline{71}$	80	2	75	$\overline{76}$	73	4,74	4,90	5,06
16	2	$\underline{80}$	$\underline{80}$	$\underline{80}$	2	$\underline{77}$	$\overline{85}$	73	5,00	5,16	5,31
17	2	82	89	$\underline{80}$	3	$\underline{86}$	85	84	4,71	4,89	5,07
18	1	$\underline{89}$	91	$\underline{89}$	1	93	87	$\overline{93}$	4,95	5,06	5,17
19	3	98	$\underline{91}$	100	2	95	$\overline{96}$	93	4,79	4,93	5,06
20	2	$\underline{100}$	$\underline{100}$	$\underline{100}$	2	$\underline{97}$	$\overline{105}$	93	5,00	5,15	5,31
21	2	102	109	$\underline{100}$	3	$\underline{106}$	105	104	4,76	4,90	5,04
22	1	$\underline{109}$	111	$\underline{109}$	1	$\underline{113}$	107	$\overline{113}$	4,97	5,05	5,14
23	3	118	$\underline{111}$	120	2	115	$\overline{116}$	113	4,83	4,94	5,04
24	2	$\underline{120}$	$\underline{120}$	$\underline{120}$	2	117	$\overline{125}$	113	5,00	5,10	5,20
25	2	122	129	$\underline{120}$	3	$\underline{126}$	125	124	4,80	4,92	5,04
26	1	$\underline{129}$	131	$\underline{129}$	1	133	127	$\overline{133}$	4,96	5,04	5,11
27	3	138	$\underline{131}$	140	2	135	$\overline{136}$	133	4,86	4,95	5,04
28	2	$\underline{140}$	$\underline{140}$	$\underline{140}$	2	$\underline{137}$	$\overline{145}$	133	5,00	5,10	5,09
29	2	142	149	$\underline{140}$	3	$\underline{146}$	145	144	4,84	4,94	5,04
30	1	$\underline{149}$	151	$\underline{149}$	1	153	147	$\overline{153}$	4,97	5,04	5,10

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Counting the number of cases of application/use with the player of each strategy and Dale him to the number of party/batches k, we will obtain approximate **values of the probabilities** with which are applied strategies in **the optimum mixture**

$$S_A^* = (p_1, p_2, p_3); \quad S_B^* = (q_1, q_2, q_3).$$

As can be seen from Table 10.1, value v^* insignificantly varies about worth of game $v = 5$ (value of initial game was 0, we added to all matrix elements on 5).

FOOTNOTE [1]. Register $P_2^* = p_2 = 0.5$ is, of course, random. ENDFOOTNOTE.

Counting according to Table 10.1 frequencies of the application/use of strategies A_1, A_2, A_3 in the first 30 party/batches, **we will obtain:**

$$P_1^* = {}^8/_{30} \approx 0{,}267; \quad P_2^* = {}^{15}/_{30} = 0{,}5; \quad P_3^* = {}^7/_{30} \approx 0{,}233.$$

They render/showed sufficiently close to the known to us from the solution of game **probabilities:**

$$p_1 = {}^1/_4 = 0{,}25; \quad p_2 = {}^1/_2 = 0{,}5; \quad p_3 = {}^1/_4 = 0{,}25^{*)}.$$

Analogously for player B we find the frequencies of strategies B_1, B_2, B_3 in the first **30 party/batches:**

$$Q_1^* = {}^6/_{30} = 0{,}2; \quad Q_2^* = {}^{15}/_{30} = 0{,}5; \quad Q_3^* = {}^9/_{30} = 0{,}3.$$

This already more intensely differs from the solution of game according to which:

$$q_1 = 0.25; q_2 = 0.5; q_3 = 0.25.$$

But for us indeed are important not precise values of probabilities q_1, q_2, q_3, but gain, which is ensured to us by the application/use of mixed strategies. If enemy will use the mixed strategy

$$\tilde{S}_B{}^* = (0,2;\ 0,5;\ 0,3).$$

Our gain (his loss) will be not greater than 5.10 (last/latter row in Table 10.1), that only a little differs from the worth of game 5.0. Let us note that by posing practical play problem, we usually make simplification and the assumptions which make excessive pursuit of the high accuracy of solution, so that the tentative solution of game, obtained by the method of iterations (even with the small number of "party/batches"), frequently can render/show sufficient.

Let us do apropos Table 10.1 still one observation. In it are encountered the rows (for example, the eighth, twelfth, twentieth and so forth), where all three values of gains are emphasized; this means that is reached the "position of equilibrium", by which any behavior of enemy gives to us one and the same gain, namely, worth of game v.

Let us focus attention on the fact that for these rows real/actually value $\underline{v}$ reaches accurately values v. According to

such sign/criteria it is possible to find approximate value of worth of game: if in some consecutive columns with all strategies of contrary side is ensured approximately one and the same gain, this means that it can be taken for approximate value of worth of game. The knowledge of approximate value of worth of game is important for during stopping the of process of iterations.

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How to find virtually optimal strategies after the process of iterations it is ended? Let us return to Table 10.1 and will consider in it column $\underline{v}$. Let us find in this column maximum cell/element. In our case this render/showed $\underline{v}_{max}$ = 5 (randomly equal worth of game, but, beginning the iterations, we indeed it not know!). From this we consist that applying the mixed strategy, which corresponds to this row, we ensure to ourselves the gain, not less than 5. Let us count the frequencies of strategies for the 20th row. Strategy A_1 was applied by us 5 times of 20, strategy A_3 - also 5 times, strategy A_2 - 10 times, whence we take the probabilities of strategies:

$$p_1 = 0{,}25, \quad p_2 = 0{,}5, \quad p_3 = 0{,}25,$$

which, as one would expect that it coincides (in this case it is accurate, but not approximately) with the optimal strategy of player A in the solution of game. Let us do the same for player B. Let us consider column $\bar{v}$ and will find in it minimum number $\bar{v}_{min}$. This will

be 5.04, attained, for example in the 29th row. This means that if player B will apply the mixed strategy, which corresponds to entire "past" for this row: q_1 = 5/29 = 0.172; q_2 = 15/29 = 0.517; q_3 = 9/29 = 0.311, then it can guarantee, which will lose no more than 5.04. This - is better than value 5.10, attained $\bar{v}$ in quite last/latter row.

Thus, even with the small number of iterations (k = 30) worth of game and solution are located with satisfactory accuracy.

11. Physical mixture of strategies.

Solving the problems of the theory of games, we repeatedly came to conclusion/derivations, the recommending players to apply not the pure/clean, but mixed strategies. Let us consider a question concerning the actual realization of mixed strategies in practice.

The basic region where is applied the theory of games - conflicting situations, connected with the combat operations where the deliberate counteraction of the reasonable enemy is not subject to doubt and always must be included in the model of operation.

The problems of operations research, connected with combat operations, it is possible to conditionally divide into two class: -

"technical" and "tactical" problems. In "technical" problems speech occurs about the selection of the rational design parameters of the specimen/samples used combat of technology. In "tactical" problems speech occurs from the methods of the combat employment of the already available technical equipment with the assigned parameters; this - more movable, is more "topical" solutions. Significant part of them will be be accepted and be justified in the course of quite combat operations.

Let us consider a question concerning the application/use of mixed strategies in each problems.

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As concerns "tactical" problems, here the applicability of the mixed strategies of doubts is not caused: they indicate the flexible, movable, always unexpected for an enemy tactics. The advisability of this tactics was obvious always; by play methods it is possible to only justify the proportions of different tactical methods.

In "technical" problems matter is somewhat otherwise. Let, for example, the speech occur from that in order to select made of several possible versions and to carry out a new specimen/sample of armament. Scarcely whether will expediently let this selection of

chance, for example toss up coin and, if falls coat of arms, select the first version, and if numeral - the second. This is inexpedient at least because the essence of mixed strategy in the fact that its concrete/specific/actual realization always remains mystery for an enemy, and when speech occurs about lasting solution, at enemy, as a rule, will be the time and possibility to gather information about strategy accepted and to act with respect to it.

In similar problems play principles can be applied otherwise: in the form of the so-called "physical mixture of strategies". The physical mixture of strategies is called such mixture with which simultaneously (in one or several operations) they are applied several strategies in the specific proportions; for example, several specimen/samples of armament, which possess different properties. If the specimen/samples used are sharply different in their characteristics, then, using physical mixture of strategies, we can noticeably increase our gain in comparison with that case when is applied only one strategy. The proportions in which must be mixed different specimen/samples, can be substantiated on the basis of the principles of the theory of games.

As the examples of the physical mixture of strategies, it is possible to give: - application/use in automatic gun of ammunition belt, complete cartridges of different types (armor-piercing,

ignition, explosive); - arrangement in the band of PVO of antiaircraft complexes with different characteristics; - application/use in the combat operations of uniform fighters with different armament, etc.

Strictly speaking, the physically mixed strategy is that not mixed, but pure/clean; its parameters are the proportions, in which are mixed separate specimen/samples. However, stated so play problem proves to be, as a rule, very complex (at least because the number of strategies in this case it is infinite). In the first approximation, it is possible to solve the problem of the establishment of these proportions on the basis of the theory of final games and by substituting the optimum mixed strategy by physical mixture. This approximate solution of the game most of all is approached in the case when the situation of the combat employment of specimen/samples of armament previously is not completely clear; in this case, the presence as arms simultaneously of several specimen/samples with different characteristics and to a certain degree ensures their application/use in combat operations in those proportions on the average, in which they are in the presence.

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Example. Available are the worked out four specimen/samples of

the AA guided missiles: A_1, A_2, A_3, A_4, intended for shooting at aircraft; are known the aircraft types of enemy B_1, B_2, B_3, B_4, B_5, which it can apply; however, it is unknown previously - in which proportion. The kill probability of the aircraft of enemy so on application/use of each type of armament is assigned by the matrix/die:

A_i \ B_j	B_1	B_2	B_3	B_4	B_5
A_1	0,2	0,4	0,6	0,4	0,7
A_2	0,3	0,4	0,6	0,5	0,8
A_3	0,4	0,5	0,6	0,5	0,8
A_4	0,7	0,3	0,5	0,2	0,1

It is required, on the basis of the principles of the theory of games, to justify the proportions in which it is necessary to order the armament of different types.

Solution. We note that strategy A_1 is knowingly unfavorable in comparison with A_2; strategy A_2, is knowingly unfavorable in comparison with A_3; game is reduced to game 2 x 5 with the matrix/die:

A_i \ B_j	B_1	B_2	B_3	B_4	B_5
A_3	0,4	0,5	0,6	0,5	0,8
A_4	0,7	0,3	0,5	0,2	0,1

Further, we note that strategy B_3 for an enemy is clearly

unfavorable in comparison with B_2, and B_2 - in comparison with B_4. Remains game 2 x 3 with the matrix/die:

A_i \ B	B_1	$B_{[illegible]}$	$B_{[illegible]}$
A_3	0,4	0,5	0,8
A_4	0,7	0,2	0,1

Let us construct the geometric interpretation of game (Fig. 9. 17).

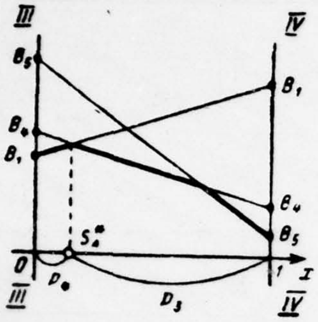

Fig. 9.17.

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From curve/graph it is evident that active strategies of enemy they are B_1 and B_4; game is reduced to game 2 x 2:

A_i \ B_j	B_1	B_4
A_3	0,4	0,5
A_4	0,7	0,2

We find the solution of the game:

$$S_A^* = (0,\ 0,\ p_3,\ p_4),$$

where

$$p_3 = \frac{0,2-0,7}{0,4+0,2-0,5-0,7} = \frac{5}{6};\quad p_4 = 1 - p_3 = \frac{1}{6};$$

$$v = \frac{0,2\cdot 0,4 - 0,5\cdot 0,7}{0,4+0,2-0,5-0,7} = \frac{9}{20} = 0,45.$$

Thus, on the basis of the principles of the theory of games, we take the recommendations: not to order at all specimen/samples A_1 and

A_2, but specimen/samples A_3, A_4 to order in proportion 5:1. In this case, the average kill probability of the aircraft of enemy (during the mass application/use of specimen/samples of armament) let us be maximum (it is not below 0.45).

12. Cell/elements of the theory of statistical solutions.

In the problems of the theory of games, examining the operations, conducted under conditions of indeterminancy/uncertainty, we they connected this indeterminancy/uncertainty with the unknown for us behavior of the enemy they proceeded from the fact that this enemy it is "reasonable and ill-intentioned" it launches those and precisely those actions which for us are least advantageous.

However, during operations research is necessary to meet not only this form of indeterminancy/uncertainty. Very frequently the indeterminancy/uncertainty, which accompanies operation, connected not with the conscious counteraction of the enemy, but it is simple from our insufficient information about the conditions under which will be carried out the operation. Thus, for instance, can be previously unknown: weather in certain region, purchasing demand on of the specific form production, the volume of transport which it is necessary to implement to railroad and, etc.

In all of this type the cases of the condition of the execution of operation depend not on consciously counteracting to us enemy, but on the objective reality which in the theory of solutions is conventionally designated as "nature". The corresponding situations are frequently called "games with nature". "nature" in the theory of statistical solutions is considered as certain disinterested instance whose "behavior" is unknown, but, in any case, does not contain the cell/element of the conscious counteraction to our plan/layouts.

Let us consider of this type situation. Let there be at us (side A) m of possible strategies: $A_1, A_2, \ldots, A_m$; as concerns situation, about it it is possible to do n of assumptions $\Pi_1, \Pi_2, \ldots, \Pi_n$ — let us consider them as "strategy of nature".

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Our gain a_{ij} with each pair of strategies A_i, Π_j is assigned by matrix/die (Table 12.1):

Table 12.1

A_i \ Π_j	Π_1	Π_2	...	Π_n
A_1	a_{11}	a_{12}	...	a_{1n}
A_2	a_{21}	a_{22}	...	a_{2n}
...	...	...	...	...
A_m	a_{m1}	a_{m2}	...	a_{mn}

It is required to select player's such a strategy A (pure/clean or mixed), which is preferable (more advantageous) in comparison with others.

At first glance it can seem that stated problem is simpler than the recreation area, since it does not contain counteraction. It is real/actual, that makes decision in game with nature more easily in that sense, that it, most likely, will be obtained in this game larger gain than in game against conscious enemy; however to it to with more difficulty make the substantiated decision which will give a good gain. The point is that in play conflicting situation the assumption about the diametric opposition of the interests of enemy by our in a sense seemingly remove/takes indeterminancy/uncertainty. But if this assumption cannot be done, indeterminancy/uncertainty manifests itself to much more powerful degree.

The simplest case of the selection of solution under conditions of indeterminancy/uncertainty is the case when some one of player's strategies A exceeds others ("prevails" above them) as, for example, it is shown in Table 12.2.

In this table gain with strategy A_2 in any state of nature Π_j

not less than gain with any other strategy; that means strategy A_2 is preferable ("prevails" above all others), and with it it should be used.

Table 12.2

A_i \ Π_j	Π_1	Π_2	Π_3	Π_4
A_1	1	2	3	5
A_2	7	4	4	5
A_3	3	4	4	1
A_4	7	4	2	2

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If in even in matrix/die there is no prevailing strategy, all the same one should look over it at the visual angle of strategies, knowingly unfavorable for a player, worse, than at least one of the others, or that duplicate, which must be reject/thrown. For example, in Table 12.3 it is possible to reject/throw strategies A_1, A_2, knowingly unfavorable in comparison with A_4, and strategy A_5 - in comparison with A_3, as a result of which matrix/die it will be reduced to matrix/die 2 x 5 (see Table 12.4).

Let us focus attention on following: in game against reasonable enemy, we would reject/throw for it strategy Π_3 as unfavorable in comparison with Π_4, and Π_4 - in comparison with Π_5; in "game against nature" this cannot be made, since "nature" does not choose its strategy (state) so that as much as possible to us "to do harm".

Subsequently we will assume that the analysis of matrix/die and the rejection of knowingly unfavorable and duplicating strategies are already produced.

How to us to be guided in the matter of making decision in the situation of indeterminancy/uncertainty, if not one strategy does prevail above others? It is clear that we must proceed from the matrix/die of gains (a_{ij}). However, sometimes the picture of situation, which gives the matrix/die of gains, contains its kind of "distortion".

Let us explain, that we keep in mind. Let us assume that the gain in strategy A_i and state of nature Π_j, is more than in strategy A_k and state of nature Π_l:

$$a_{ij} > a_{kl}.$$

But the first gain can be more than the second not because we selected more successful strategy, but it is simple because the state of nature Π_j "is more advantageous" for us than state Π_l.

Table 12.3

A_i \ Π_j	Π_1	Π_2	Π_3	Π_4	Π_5
A_1	5	3	4	2	1
A_2	5	3	2	1	1
A_3	1	2	5	4	3
A_4	7	6	7	3	1
A_5	1	2	3	4	3

Table 12.4

A_i \ Π_j	Π_1	Π_2	Π_3	Π_4	Π_5
A_3	1	2	5	5	3
A_4	7	6	7	3	1

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For example, for any economic operation state the "absence of the natural calamities" generally is more favorable than state "flood", "earthquake", etc. It is represented desirable to introduce such indices which would not simply give gain in each situation, but would be described "success" or "failure" of the application/use of datum of strategy in this situation, taking into account that, how generally this situation was favorable for us.

For this purpose, in the theory of solutions, is introduced the important concept of "risk".

The risk of the player during the use of strategy A_i under conditions Π_j is called the difference between the gain, which he would obtain, if he would know that Π_j, and by the gain which it will obtain under the same conditions, applying strategy A_i.

Let us designate r_{ij} player's risk with his strategy A_i under conditions Π_j. Is expressed risk r_{ij} through the matrix elements of gains (a_{ij}). It is obvious, if player knew previously the state of nature (condition) Π_j, he would select the strategy to which corresponds the maximum gain in this chair, it is shorter, the "maximum of column" - let us designate it, as earlier, β_j. According to the determination of risk,

$$r_{ij} = \beta_j - a_{ij}, \tag{12.1}$$

where $\beta_j = \max\limits_i a_{ij}$.

From this determination it follows that the risk cannot be negative:

$$r_{ij} \geqslant 0.$$

During the calculation of the risk, which corresponds to each strategy under given conditions, is considered common/general/total favorableness or unfavorableness for us of this state of the nature: value β_j serves as if criterion/standard of favorableness of state.

The matrix/die of risks (r_{ij}) gives the often more demonstrative picture of the indefinite situation, than the matrix/die of gains (a_{ij}).

Example 1. The operation is planned under the previously unclear conditions, which concern, for example, market conjuncture. Relative to these conditions it is possible to do different assumptions:. $\Pi_1, \Pi_2, \Pi_3, \Pi_4$.

The advantage of operation (expected gain) with our strategies (A_i) for varied conditions (Π_j) is assigned by the matrix/die of gains (a_{ij}) (Table 12.5).

Table 12.5

A_i \ Π_j	Π_1	Π_2	Π_3	Π_4
A_1	1	4	5	9
A_2	3	8	4	3
A_3	4	6	6	2

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To construct the matrix/die of risks (r_{ij}).

Solution. Each matrix element is subtracted from maximum in this column value (in the first column this $\beta_1 = 4$, in the others $\beta_2 = 8$, $\beta_3 = 6$, $\beta_4 = 9$. We obtain the matrix/die of risks (Table 12.6).

Tables 12.6

A_i \ Π_j	Π_1	Π_2	Π_3	Π_4
A_1	3	4 •	1	0
A_2	1	0	2	6
A_3	0	2	0	7

With view on this matrix/die to us become clearer some features of this "game with nature". So, in matrix/die from gains (a_{ij}) (see Table 12.5) in the second row the first and last/latter cell/elements were equal to each other: $a_{21} = a_{24} = 3$.

However, these gains are entirely nonequivalent to each other in the sense of that, is how successfully selected strategy: in the state of nature Π_1 we could win larger anything 4, and the selection of strategy A_2 is almost completely good; but thus in state Π_4 we could, after selecting strategy A_1, to win on whole 6 ones is more, i.e., the selection of strategy A_2 very poor. This is reflected by the matrix elements of the risks: $r_{21} = 1$, $r_{24} = 6$.

That that we made, until now, - altogether only different

methods of the grouping of initial data; as concerns criteria for making of decisions, we them will consider in following paragraph.

13. Criteria, based on the known probabilities of conditions. Criteria of Wald, Hurwitz, Savage.

Most simply is solved the problem of the selection of solution under conditions of the indeterminancy/uncertainty when to us although are unknown the conditions of the execution of operation (state of nature) $\Pi_1, \Pi_2, \ldots, \Pi_n$, but are known to their probability:

$$Q_1 = P(\Pi_1);\quad Q_2 = P(\Pi_2);\quad \ldots;\quad Q_n = P(\Pi_n)$$

$$\left(\sum_{j=1}^{n} Q_j = 1\right).$$

In this case as the index of efficiency which we attempt to turn into maximum, it is logical to take average value, or the mathematical expectation of gain, taking into account the probabilities of all possible conditions.

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Let us designate this average value for player's i strategy through $\bar{a}_i$:

$$\bar{a}_i = Q_1 a_{i1} + Q_2 a_{i2} + \ldots + Q_n a_{in},$$

or, it is shorter,

$$\bar{a}_i = \sum_{j=1}^{n} Q_j a_{ij}. \tag{13.1}$$

It is obvious, $\bar{a}_i$ to eat nothing else but is weighted mean the gains of the i row, undertaken with weights Q_1, Q_2, ..., Q_n. As the optimal strategy it is logical to select that from strategies $A^* = A_i$, for which value $\bar{a}_i$ is converted into maximum.

With the help of this method the problem of the selection of solution under conditions of indeterminancy/uncertainty is converted into the problem of the selection of solution under conditions of definition, only taken solution is optimum not in each individual case, but on the average.

Example 1. Plan/glides operation under previously unknown meteorological conditions; the versions of these conditions: Π_1, Π_2, Π_3, Π_4. According to the materials of weather reports after many years of the frequency (probability) of these versions, are equal respectively:. $Q_1 = 0,1;\ Q_2 = 0,2;\ Q_3 = 0,5;\ Q_4 = 0,2$.

The possible versions of the organization of operation in different meteorological conditions yield different advantage. The values of "income" for each solution under different conditions are given in Table 13.1.

Table 13.1

A_i \ Π_j	Π_1	Π_2	Π_3	Π_4	$\bar{a}_i$
A_1	1	4	5	9	5,2*
A_2	3	8	4	3	4,5
A_3	4	6	6	2	5,0
Q_j	0,1	0,2	0,5	0,2	

In last/latter row are given to the probability of conditions. Average gains $\bar{a}_i$ are given in last/latter column. From it it is evident that the optimal strategy of player is his strategy $A^* = A_1$, which gives the average gain $\bar{a}_1 = 5.2$ (it is noted by asterisk).

When selecting of the optimal strategy under unknown conditions with known probabilities, it is possible to use only average gain

$$\bar{a}_i = \sum_{j=1}^{n} Q_j a_{ij},$$

but also by the average risk

$$\bar{r}_i = \sum_{j=1}^{n} Q_j r_{ij},$$

which, it goes without saying, must be converted not into maximum, but into the minimum.

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Let us show that strategy, which maximizes average gain $\bar{a}_i$,

coincides with strategy, which minimizes average risk $\bar{r}_i$. Let us compute both these of index and will add them:

$$\bar{a}_i+\bar{r}_i=\sum_{i=1}^{n} Q_j a_{ij}+\sum_{i=1}^{n} Q_j(\beta_j-a_{ij})=$$

$$=\sum_{i=1}^{n} Q_j a_{ij}-\sum_{i=1}^{n} Q_j a_{ij}+\sum_{i=1}^{n} Q_j \beta_i=\sum_{i=1}^{n} Q_j \beta_j. \qquad (13.2)$$

This sum (weighted mean value of the maximums of columns) for this matrix/die is a constant value. Let us designate its C:

$$\sum_{i=1}^{n} Q_j \beta_j=C.$$

Then

$$\bar{a}_i+\bar{r}_i=C,$$

whence average risk is equal.

$$\bar{r}_i=C-\bar{a}_i. \qquad (13.3)$$

It is obvious, this value is converted into the minimum at the same time, when $\bar{a}_i$ - into maximum, therefore, strategy, selected from the conditions of minimum average risk, coincides with strategy, selected from the conditions of maximum average gain.

Let us note that in the case when are known to the probability of the states of nature Q_1, Q_2, ..., Q_n, during the solution of game with nature always it is possible to manage with some pure strategies, without applying those mixed. It is real/actual, if we are apply some mixed strategy

$$S_A=(p_1, p_2, \ldots, p_m),$$

i.e. strategy A_1 with probability p_1, strategy A_2 with probability p_2

and so forth, then our average gain, averaged and according to conditions (states of nature) and on our strategies, will be:

$$\bar{a}=p_1\bar{a}_1+p_2\bar{a}_2+\ \dots\ +p_m\bar{a}_m.$$

This - is weighted mean the gains $\bar{a}_i$, which correspond to our pure strategies.

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But it is clear that any average cannot exceed maximum from the averaged values:

$$\bar{a}\leqslant \max_i \bar{a}_i.$$

Therefore the application/use of mixed strategy S_A with any probabilities p_1, p_2, ..., cannot be more advantageous for a player than the application/use of pure strategy A*.

The probabilities of conditions (states of nature) Q_1, Q_2, ..., Q_n can be determined from statistical data, connected with repeated execution similar operations or simply with conducting of observations on by the properties of nature. For example if railroad for this time interval one must fulfill not the completely known volume of transport, then the data on the distribution of conditions can be undertaken from an experiment in the past years. If, as in the previous example, the success of operation depends on meteorological conditions, the data on them can be undertaken from the statistics of

weather reports.

However, frequently are encountered the cases when, beginning the execution of operation, we do not have an idea of the probabilities of the states of nature; all the our information leads to the enumeration of the versions of states, and to consider their probabilities we not can. Thus, for instance, scarcely whether to us be managed to reasonable consider probability that during nearest k of years will be suggested and realized the important technical invention.

It goes without saying that in the similar cases of the probability of conditions (states of nature) they can be estimated subjectively: some of them are represented to us more, and others - by less plausible. In order our subjective representations of greater or the smaller "likelihood" of one or the other hypothesis to convert into numerical estimations, can be applied different technical methods. So, if we cannot prefer one hypothesis, if they everything for us are equal, then it is logical to assign their probabilities equal each other:

$$Q_1 = Q_2 = \ldots = Q_n = 1/n.$$

This - the so-called "principle of Laplace's insufficient basis/base". Another frequently encountered case - when we have a representation of which conditions are more probable, but which -

less, i.e., we can place the available hypotheses in descending order of their likelihood: in all is plausible the first hypothesis (Π_1), then the second (Π_2) and so forth; least of all is plausible the n hypothesis (Π_n). However, how one of them more probable another - we do not know. In this case it is possible, for example to assign the probabilities of hypotheses proportional to the terms of the decreasing arithmetical progression:

$$Q_1 : Q_2 : \dots : Q_n = n : (n-1) \quad \dots : 1,$$

or, taking into account that $Q_1+Q_2+ \dots +Q_n=1$,

$$Q_i = \frac{2(n-i+1)}{n(n+1)} \quad (i = 1, 2, \dots, n).$$

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Sometimes it succeeds, on the basis of experiment and the common sense, to consider fine/thinner differences between the degrees of the plausibility of hypotheses.

Similar methods of the subjective evaluation "probability-likelihood" of different hypotheses about the state of nature can sometimes aid when selecting of solution. However, it cannot be forgotten which "optimum solution", selected on the basis of subjective probabilities, also will unavoidably render/show subjective. The degree of the subjectivity of solution can be decreased, if we instead of probabilities Q_1, Q_2, ..., Q_n, assigned

arbitrarily by one person, introduce average of such probabilities, assigned, it is independent of each other, by the group of the qualified persons ("experts"). The method of experts's interrogation is generally widely applied in contemporary science, when speech occurs about the evaluation of the indefinite situation (for example, in futurology). An experiment in the application/use of similar methods teaches that often the estimations of the experts (accepted independently one of another) prove to be by no means so/such contradictory, as this it was possible to assume previously, and to deduce of them some prerequisite/premises for making of reasonable decision is completely possible.

Above we threw light a question concerning the selection of solution on the basis of the objectively calculated or subjectively assigned probabilities of the states of nature. This approach in the theory of solutions - not only. Besides it there exist still several of the "criteria" or of approaches to the selection of optimum solution under conditions of indeterminancy/uncertainty. Let us pause at some of them.

1. Maximin Wald criterion.

According to this criterion as optimum, is chosen player's that strategy A, by which minimum gain is maximum, i.e., strategy, which

guarantees under any conditions the gain, not less than the maximin:

$$\mathcal{W} = \max_i \min_j a_{ij}. \qquad (13.4)$$

If we are guided by this criterion, necessary always to be oriented toward the worse conditions and to choose TU strategy, for which in the worst conditions the gain is maximum. Using this criterion in games with nature, we seemingly place of instead of this impersonal and disinterested instance active and ill-intended enemy. It is obvious, this approach can be dictated only by extreme pessimism in the evaluation of situation - "it is always necessary to rely on is worse" ! -, but as one of the possible approaches it merits consideration.

2. Criterion of the minimax risk of Savage.

This criterion recommends under conditions of indeterminancy/uncertainty to choose ~~TU~~ that one strategy with which the value of risk takes small value in most unfavorable situation (when risk is maximum):

$$\mathcal{S} = \min_i \max_j r_{ij}. \qquad (13.5)$$

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Essence of this criterion in, by any ways avoiding the large risk with the taking of solution.

The criterion of Savage, just as the criterion of Wald - this the criterion of extreme pessimism, but only pessimism is here understood differently: worse declares not minimum gain, but the maximum loss of gain in comparison with the fact that it would be possible to achieve under given conditions (maximum risk).

3. Criterion of Hurwitz' pessimism-optimism.

This criterion is recommended under conditions of indeterminancy/uncertainty when selecting of solution not to be guided either by the extreme pessimism (always rely on the worst!) or by the extreme, frivolous optimism (continually will manage best!) Hurwitz' criterion it takes the form:

$$\mathcal{H} = \max_i \left\{ \varkappa \min_j a_{ij} + (1-\varkappa) \max_j a_{ij} \right\}, \qquad (13.6)$$

where $\varkappa$ — the coefficient, selected between zero and one.

We analyze the structure of expression (13.6). When $\varkappa = 1$ criterion of Hurwitz is converted into pessimistic criteria of Wald, while with $\varkappa = 0$ - into the criterion of "extreme optimism", which recommends to choose ~~ee~~ that one strategy, for which under the best conditions the gain is maximum. With $0 < \varkappa < 1$ it is obtained something average among extreme pessimism and the extreme optimism (coefficient

× expresses as the "measure of researcher's pessimism"). This coefficient is chosen of the subjective considerations - than more dangerous situation, the more we wish in it " to fear", the nearer to one is chosen ×.

With wish it is possible to construct the criterion, analogous to the criterion of Hurwitz' optimism-pessimism by proceeding not from gain, but from scratch as in the criterion of Savage, but we on this will not be stopped.

Despite the fact that the selection of criterion as identification of parameter in Hurwitz' criterion, they are subjective, all the same it can render/show on desire to look over situation from the point of these criteria. If the recommendations, which result from different criteria, coincide - so much the better, it is possible to boldly choose recommended with them solution. But if, as so often is the case, recommendations contradict each other - it always makes sense to be planned above this and to make final decision taking into account his powerful and weak sides. The analysis of the matrix/die of game with nature at the visual angle of different criteria frequently gives better/best representation of situation, of advantages and disadvantages in each solution, than the direct examination of matrix/die, especially, when its size/dimensions are great.

Example 2. Is examined game with nature 4 x 3 with player's four strategies: A_1, A_2, A_3, A_4 and by three versions of conditions (states of nature): Π_1, Π_2, Π_3. The matrix/die of gains is given in Table 13.2.

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To find optimum solution (strategy), using the criteria of Wald and Savage and Hurwitz' criterion when $\varkappa = 0,6$.

Solution. 1. Criterion of Wald.

In each matrix row, we take the smallest gain (Table 13.3).

From value α_i maximum (it is noted by asterisk) is equal to 0.25, therefore, on the criterion of Wald optimum is strategy A_3.

2. Criterion of Savage.

We construct the matrix/die of risks and we place in right additional column the maximum risk in each row γ_i (Table 13.4).

Minimum from values γ_i it is 0.60 (it is noted by asterisk); consequently, on the criterion of Savage, optimum is any of strategies A_2, A_3.

3. Criterion of Hurwitz $(x = 0,6)$.

We record/write in the right three columns of matrix/die (table 13.5) the "pessimistic" estimation of gain α_i, "optimistic" ω_i and them weighted mean according to formula (13.6):

$$h_i = 0,6\alpha_i + 0,4\omega_i.$$

Maximum value h_i (noted by asterisk) corresponds to strategy A_3. Consequently, on Hurwitz' criterion with light/lung preponderance to the side of pessimism $(x = 0.6)$ the optimal strategy is A_3. Thus, all the three criteria accordingly speak in favor of strategy A_3, which we have all foundations for selecting.

Table 13.2.

A_i \ Π_j	Π_1	Π_2	Π_3
A_1	0,20	0,30	0,15
A_2	0,75	0,20	0,35
A_3	0,25	0,80	0,25
A_4	0,85	0,05	0,45

Table 13.3.

A_i \ Π_j	Π_1	Π_2	Π_3	α_i
A_1	0,20	0,30	0,15	0,15
A_2	0,75	0,20	,35	0,20
A_3	0,25	0,80	0,25	0,25*
A_4	0,85	0,05	0,45	0,05

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Table 13.4.

Π_j / A_i	Π_1	Π_2	Π_3	γ_i
A_1	0,65	0,50	0,30	0,65
A_2^*	0,10	0,60	0,10	0,60*
A_3^*	0,60	0	0,20	0,60*
A_4	0	0,75	0	0,75

Table 13.5.

Π_j / A_i	Π_1	Π_2	Π_3	α_i	ω_i	h_i
A_1	0,20	0,30	0,15	0,15	0,30	0,21
A_2	0,75	0,20	0,35	0,20	0,75	0,42
A_3^*	0,25	0,80	0,25	0,25	0,80	0,47*
A_4	0,85	0,05	0,45	0,05	0,85	0,37

Table 13.6.

Π_j / A_i	Π_1	Π_2	Π_3	Π_4
A_1	2	3	4	5
A_2	5	4	1	2
A_3	7	2	8	1

Table 13.7.

A_i \ Π_j	Π_1	Π_2	Π_3	Π_4	α_i
A_1^*	2	3	4	5	2*
A_2	5	4	1	2	1
A_3	7	2	8	1	1

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Example 3. Is examined game against nature 3 x 4 with the matrix/die of gains, placed in Table 13.6.

To select **the optimal str**ategy on the criteria of Wald, Savage, and Hurwitz when $\varkappa = 0,5$

Solution 1. Criterion of Wald (Table 13.7).

Optimal strategy A_1.

2. Criterion of Savage (Table 13.8).

Optimal strategy A_3.

3. Hurwitz' criterion $(\varkappa = 0,5)$ (Table 13.9).

Optimal strategy A_3.

Thus, the criteria of Savage and Hurwitz speak in favor of strategy A_3, whereas the criterion of Wald is recommended A_1. If of that making decision there are no special basis/bases it stops to the point of extreme pessimism, it is possible to stop at strategy A_3.

In conclusion let us note following. All the three criteria - Wald, Savage, and Hurwitz - have formulated [redacted] for the pure strategies; but completely thus it is possible to form them, also, for mixed strategies. For example, according to the criterion of Savage one should choose that [redacted] mixed strategy

$$S_A = (p_1, p_2, \ldots, p_m),$$

for which it is reached

$$\min_{(p_1, p_2, \ldots, p_m)} \max_i (p_1 r_{1j} + p_2 r_{2j} + \ldots + p_m r_{mj})$$

(minimum is taken on all $p_1, p_2, \ldots, p_m \geqslant 0, p_1 + p_2, + \ldots + p_m = 1$). To find this minimax (or maximin in the criterion of Wald) is possible the usual methods of linear programming.

There can be the cases, when application/use of mixed strategies with the use of the criteria of Wald, Savage, and Hurwitz will give advantage in comparison with that solution where are applied some pure strategies; however, we will examine these criteria only for the pure strategies.

Table 13.8.

A_i \ Π_j	Π_1	Π_2	Π_3	Π_4	γ_i
A_1	5	1	4	0	5
A_2	2	0	7	3	7
A_3^*	0	2	0	4	4*

Table 13.9.

A_i \ Π_j	Π_1	Π_2	Π_3	Π_4	α_i	ω_i	h_i
A_1	2	3	4	5	2	5	3,5
A_2	5	4	1	2	1	5	3,0
A_3^*	7	2	8	1	1	8	4,5*

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One of the reasons for this - the fact that we wish to avoid complex calculations, when their result can be reduced on no by deficiency/lack in the information about situation (not knowing the probabilities of conditions). Another, more important reason - in the fact that the basic content of the theory of the statistical solutions (we will touch it in following paragraph) - this gliding/planning of obtaining and use of further information about

the state of the nature which can be extracted via experiment. Research shows that in the typical cases when speech occurs about obtaining of any considerable quantity of further information, the criteria, which do not use the probabilities of states (Wald, etc.), become virtually equivalent to the criterion, based on the probabilities of states. But we know that with the use of this criterion the application/use of mixed strategies does not have sense; therefore, if we can obtain as much as possible further information, the application/use of mixed strategies becomes meaningless (whatever from the criteria of the selection of solution we used). But if we not can running experiments, to obtain new information, then different criteria can give which contradict each other recommendations, as we saw in example [redacted] 3.

14. Planning experiment under conditions of indeterminancy/uncertainty.

In this paragraph we will touch a very important in the theory of statistical solutions question concerning how to us can aid in making of decision the experiments, launched for purpose of the explanation of real situation? This question is central in theory, as is shown name itself: indeed word "statistical" and is used, when speech occurs about conclusion/derivations from experiments, about

their gliding/planning and treatment.

The appropriate theory can be developed both on the basis of the known probabilities of the states of nature and from the criteria, similar to the criterion of Wald; we will here examine the theory, which proceeds from the known probabilities of the states of nature as simpler.

Let us consider a following question. Us must launch certain operation in insufficiently explained conditions. Does have sense for refining the conditions in our indefinite situation to launch certain experiment $\mathcal{E}$? It is logical, this question arises only if expenditures on experiment are essential and congruent with that increase in the gain which we can obtain, after learning situation more accurately. But if expenditures on experiment are negligible, answer/response to this question always positive.

Let us consider first the case of "ideal" experiment $\mathcal{E}$, of the leading to completely precise knowledge that state of nature Π_j, which occurs in this situation.

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Let be assigned the matrix/die of gains $\|a_{ij}\|$ $(i = 1, \ldots, m;$ $j = 1, \ldots, n)$,

and, furthermore, are known to probability Q_1, Q_2, ..., Q_n varied conditions $P_1, P_2, \ldots P_n$ [П = P]. Let the expenditures on experimentation $\mathscr{E}$ be equal C. Let us compare our average gain without experimentation $\mathscr{E}$ and average gain with conducting of this experiment.

As we saw in §13, if we carry out additionally no experiment, then it is necessary as the solution to select that strategy $A^* = A_i$, for which is reached the maximum average gain:

$$\max_i \{Q_1 a_{i1} + Q_2 a_{i2} + \ldots + Q_n a_{in}\}. \qquad (14.1)$$

This there will be our gain without experimentation $\mathscr{E}$.

Now let us assume that we ran experiment $\mathscr{E}$ and they explained, which of states $P_1, P_2, \ldots, P_n$ is the real state of nature. If this render/showed P_1, then we must apply that strategy A_i, for which it is reached

$$\max_i a_{i1} = \beta_1,$$

and our gain will be equal to β_1; if the real state of nature render/showed P_2 our gain will be β_2, and so forth. Generally, in the real state of nature P_j our gain will be equal to maximum gain in the j^th column:

$$\max_i a_{ij} = \beta_j.$$

But we should previously solve, will we run experiment $\mathscr{E}$ or

not; to us it is unknown, which of states P_j in reality occurs and which there will be our gain β_j. Therefore is averaged this gain with the weights, equal to probabilities Q_j:

$$Q_1\beta_1+Q_2\beta_2+ \dots +Q_n\beta_n.$$

Taking into account the cost/value of the experiment (which must be subtracted from gain) our average gain with the application/use of ideal experiment $\mathscr{E}$ is equal to

$$Q_1\beta_1+Q_2\beta_2+ \dots +Q_n\beta_n-C. \tag{14.2}$$

Thus, we must carry cut experiment, if value (14.2) is more than (14.1); but if, nonrevolution, value (14.1) are more, then experiment $\mathscr{E}$ to us is not necessary.

It is possible to somewhat modify this rule, after making it simpler. We saw that experiment $\mathscr{E}$ to us was useful (i.e. "on means"), if

$$\max_i \{Q_1a_{i1}+Q_2a_{i2}+ \dots +Q_na_{in}\} < \\ < Q_1\beta_1+Q_2\beta_2+ \dots +Q_n\beta_n-C. \tag{14.3}$$

Let us move C into left side, and "maximum" from left side into right, after changing the sign before the sum and substituting "maximum" by the "minimum"; condition (14.3) it will be rewritten in the form: $C<\min_i \{Q_1(\beta_1-a_{i1})+Q_2(\beta_2-a_{i2})+ \dots +Q_n(\beta_n-a_{in})\}$

or, it is shorter

$$C<\min\left\{\sum_{j=1}^{n} Q_j(\beta_j-a_{ij})\right\}. \tag{14.4}$$

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But $\beta_j - a_{ij}$ there is nothing else but risk r_{ij}, but the sum of right side - average expected risk:

$$\bar{r}_i=\sum_{j=1}^{n} Q_j r_{ij}.$$

Therefore the rule of solution to the execution of experiment $\mathcal{E}$ acquires following form.

Experiment $\mathcal{E}$ must be carried out, if expenditures on its realization are lesser than the minimum average risk:

$$C<\min\bar{r}_i. \tag{14.5}$$

Otherwise one should restrain from experiment, and to use that strategy A* for which is reached this minimum of average risk.

Example 1. Is examined game with nature 3 x 4 whose conditions are given in Table 14.1 (this matrix/die we already examined into §13).

The probability of the states of nature P_1, P_2, P_3, P_4 are equal respectively: $Q_1 = 0.1$, $Q_2 = 0.2$, $Q_3 = 0.5$, $Q_4 = 0.2$.

To determine, is appropriate the "ideal" experiment whose cost/value (in those ones, in which it is expressed gain) it is equal to 2.

Solution. We pass from the matrix/die of gains to the matrix/die of risks (Table 14.2).

In right further column are written the values of average risk. Minimum from these values is equal to 1.6; consequently, experimentation with the cost/value of 2 ones is inexpedient.

Table 14.1.

A_i \ Π_j	Π_1	Π_2	Π_3	Π_4
A_1	1	4	5	9
A_2	3	8	4	3
A_3	4	6	6	2

Table 14.2.

A_i \ Π_j	Π_1	Π_2	Π_3	Π_4	$\bar{r}_i$
A_1	3	4	1	0	1,6
A_2	1	0	2	6	2,3
A_3	0	2	0	7	1,8

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Above we considered the case of "ideal" experiment $\mathcal{E}$, as a result of which the situaticn completely is clarified.

Let us now consider the case not of ideal experiment $\mathcal{E}$, which does not lead to explanation in the accuracy of the state of nature P_j, but it only gives scme indirect evidence in favor of one or the other states. In general view we can tc assume that experiment leads to appearance of one of k of the antithetical events

$$B_1, B_2, \ldots B_k,$$

moreover the probability of these events (issues of experiment) they

depend on the conditions under which it is conducted: P_1, P_2, ... or P_n. Let us designate the conditional probability of appearing event B_l under conditions P_j through

$$P(B_l/\Pi_j), \quad (j=1, 2, \ldots, n;\ l=1, 2, \ldots, k)$$

and let us consider that all these conditional probabilities to us are known.

After the realization of experiment $\mathcal{E}$, which gave issue B_l, for us it is necessary to reexamine the probabilities of the conditions: the states of nature P_1, P_2, ..., P_n will be characterized not by the previous (a priori) probabilities

$$Q_1, Q_2, \ldots, Q_n,$$

while by the new, "a posteriori" probabilities of the states:

$$\tilde{Q}_{1l}, \tilde{Q}_{2l}, \ldots, \tilde{Q}_{nl},$$

i.e. by the conditional probabilities of states P_1, P_2, ..., P_n when the experiment gave result B_l. These a posteriori probabilities are counted according to known Bayes' formula:

$$\tilde{Q}_{jl} = \frac{Q_j P(B_l/\Pi_j)}{\sum_{j=1}^{n} Q_j P(B_l/\Pi_j)}, \quad (j=1, \ldots, n) \qquad (14.6)$$

(with this and is connected the fact, that the corresponding approach to making of decision in the situation of indeterminancy/uncertainty is called Bayes).

Since the a priori probabilities of the states of nature Q_1, Q_2,

..., Q_n are substituted new, a posteriori $\tilde{Q}_{1l}, \tilde{Q}_{2l}, \ldots, \tilde{Q}_{nl}$ and also, therefore, optimal strategy A* in the general case will be replaced by new optimal strategy $\tilde{A}^*_l$, calculated taking into account a posteriori probabilities (under the condition of event

Example 2. Under conditions of example 1 with the a priori probabilities of the conditions B_l).

$$Q_1=0{,}1 \quad Q_2=0{,}2, \quad Q_3=0{,}5, \quad Q_4=0{,}2$$

is run experiment $\mathcal{E}$, employed for refining the situation. This experiment, generally speaking, can have three possible issues:

$$B_1, B_2, B_3$$

The conditional probabilities of these issues $P(B_l/\Pi_j)$ for the different states of nature P_1, P_2, P_3, P_4 are given in the matrix/die of conditional probabilities (Table 14.3).

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It is known that in experiment $\mathcal{E}$ occurred issue B_1. To compute a posteriori probabilities $\tilde{Q}_{j1}$. To indicate the new optimal strategy A^*_1.

Solution. On formula (14.6) we have:

$$\tilde{Q}_{11}=\frac{0,1\cdot 0,2}{0,1\cdot 0,2+0,2\cdot 0,9+0,5\cdot 0,4+0,2\cdot 0,3}\approx 0,043,$$

$$\tilde{Q}_{21}=\frac{0,18}{0,46}\approx 0,392;\quad \tilde{Q}_{31}=\frac{0,20}{0,46}\approx 0,435;$$

$$\tilde{Q}_{41}=\frac{0,06}{0,46}\approx 0,130.$$

Let us compute average gains $\tilde{\bar{a}}^{(1)}$ with each strategy taking into account the obtained a posteriori probabilities (Table 14.4). In last/latter table row, are placed a posteriori probabilities, in right, further column - average gains at the new values of the probabilities of states, calculated according to the formula

$$\tilde{\bar{a}}_i^{(1)}=\tilde{Q}_{11}a_{i1}+\tilde{Q}_{21}a_{i2}+\tilde{Q}_{31}a_{i3}+\tilde{Q}_{41}a_{i4}.$$

Values $\tilde{Q}_{i1}$ are given in lower table row.

Thus, taking into account result B_1 of experiment $\mathcal{E}$, the optimal strategy will be no longer A_1, but A_2.

It is certain, in order previously to solve, it is worth to us carrying out experiment $\mathcal{E}$ or not, it is necessary to previously produce similar calculations not only for one issue B_1, but also for all others. Let us continue the examination of examples.

Example 3. Under conditions of examples 1 and 2 to manufacture the rule of the solution which would be indicated with which issue of experiment which strategy to choose. To explain, how average a gain during the execution of experiment $\mathcal{E}$ is greater than average gain without the execution of this experiment.

Table 14.3.

B_i \ $П_j$	$П_1$	$П_2$	$П_3$	$П_4$
B_1	0,2	0,9	0,4	0,3
B_2	0,1	0,1	0,5	0,3
B_3	0,7	0	0,1	0,4

Table 14.4.

A_i \ $П_j$	$П_1$	$П_2$	$П_3$	$П_4$	$\tilde{\tilde{a}}_i^{(1)}$
A_1	1	4	5	8	4,96
A_2^*	3	8	4	3	5,20*
A_3	4	6	6	2	5,09
$\tilde{Q}_{j1}$	0,043	0,392	0,435	0,130	

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Solution. Let us compute the remaining a posteriori probabilities of all states of nature $\tilde{Q}_{j2}, \tilde{Q}_{j3}$ (j = 1, 2, 3, 4) when the experiment gave issues B_2, B_3 respectively calculation let us produce on the same formula (14.6):

Let us reduce all the new (a posteriori) probabilities of the states of nature with each of the issues B_1, B_2, B_3 in Table 14.5.

$$\tilde{Q}_{12}=\frac{0,1\cdot 0,1}{0,1\cdot 0,1+0,2\cdot 0,1+0,5\cdot 0,5+0,2\cdot 0,3}\approx 0,029$$

$$\tilde{Q}_{22}=\frac{0,02}{0,34}\approx 0,059;\ \tilde{Q}_{32}=\frac{0,25}{0,34}\approx 0,735;\quad \tilde{Q}_{42}=\frac{0,06}{0,34}\,0,177;$$

$$\tilde{Q}_{13}=\frac{0,1\cdot 0,7}{0,1\cdot 0,7+0,2\cdot 0+0,5\cdot 0,1+0,2\cdot 0,4}\approx 0,35.$$

$$\tilde{Q}_{23}=\frac{0}{0,20}=0;\quad \tilde{Q}_{33}=\frac{0,05}{0,25}=0,25;\quad \tilde{Q}_{43}=\frac{0,08}{0,20}=0,40.$$

Table 14.5.

B_i \ Π_j	Π_1	Π_2	Π_3	Π_4
B_1	0,043	0,392	0,435	0,130
B_2	0,029	0,059	0,735	0,177
B_3	0,350	0	0,250	0,400

Table 14.6.

A_i \ Π_j	Π_1	Π_2	Π_3	Π_4	$\tilde{\tilde{a}}_i^{(2)}$
A_1^*	1	4	5	9	5,53*
A_2	3	8	4	3	4,03
A_3	4	6	6	2	5,23
$\tilde{Q}_{j2}$	0,029	0,059	0,735	0,177	

Table 14.7.

A_i \ Π_j	Π_1	Π_2	Π_3	Π_4	$\tilde{\tilde{a}}_i^{(3)}$
A_1^*	1	4	5	9	5,20*
A_2	3	8	4	3	3,25
A_3	4	6	6	2	3,70
$\tilde{Q}_{j3}$	0,350	0	0,250	0,400	

Now for each of the events B_2, B_3 (for B_1 we already this did) let us find average gain, averaging it with the weights, equal to new, a posteriori probabilities the optimal strategy we note by

asterisk. The results of calculations for events B_2 and B_3 are respectively given in Tables 14.6 and 14.7. In the lower row table, gives the a posteriori probabilities of states, while in right column - average gains.

Now, on basis of Tables 14.4, 14.6, and 14.7 we can formulate the rule of the solution:

If experiment $\mathcal{E}$ gave result B_1 - to apply to strategy A_2; if it gave not B_1 (i.e. B_2 or B_3) - to apply strategy A_1. In this case, if experiment gave issue B_1, our average gain will be equal to 5.20; if B_2 - 5.53, and if B_3, then 5.20.

The average value of average gain with this rule of solution can be calculated as follows: let us find the composite probability of event B_1:

$$P(B_1) = Q_1P(B_1/\Pi_1) + Q_2P(B_1/\Pi_2) + Q_3P(B_1/\Pi_3) + Q_4P(B_1/\Pi_4) = 0{,}1\cdot 0{,}2 + 0{,}2\cdot 0{,}9 + 0{,}5\cdot 0{,}4 + 0{,}2\cdot 0{,}3 = 0{,}46;$$

We analogously find the probabilities of events B_2 and B_3:

$$P(B_2) = Q_1P(B_2/\Pi_1) + Q_2P(B_2/\Pi_2) + Q_3P(B_2/\Pi_3) + Q_4P(B_2/\Pi_4) = 0{,}1\cdot 0{,}1 + 0{,}2\cdot 0{,}1 + 0{,}5\cdot 0{,}5 + 0{,}2\cdot 0{,}3 = 0{,}34,$$
$$P(B_3) = Q_1P(B_3/\Pi_1) + Q_2P(B_3/\Pi_2) + Q_3P(B_3/\Pi_3) + Q_4P(B_3/\Pi_4) = 0{,}1\cdot 0{,}7 + 0{,}2\cdot 0 + 0{,}5\cdot 0{,}1 + 0{,}2\cdot 0{,}4 = 0{,}20$$

Full/total/complete average gain with this rule of solution will be:

$$\tilde{\bar{a}}^* = 0,46 \cdot 5,20 + 0,34 \cdot 5,53 + 0,20 \cdot 5,20 = 5,345$$

Let us compare this gain with the fact which we would obtain in the absence of the experiment (see example 1 §13). There we obtained $\bar{a}^* = 5.20$. Thus, the execution of experiment increased our average gain by 5.345-5.20 = 0.145. Hence follows the conclusion: if the cost/value of experiment is less than 0.145 - inexpediently.

The calculations of the advisability of experimentation, it goes without saying, can be produced proceeding not from average gain, but from average risk; in this case, will be obtained the same results.

Analogously it is possible previously to read aloud, it is profitable to us several times to lead experiment $\mathcal{E}$. It is real/actual, let, let us say, that there is possibility to produce two independent repetitions $\mathcal{E}_1$; $\mathcal{E}_2$ of experiment $\mathcal{E}$, which is characterized by the conditional probabilities of the issues: $P(B_l/P_j)$ ($j = 1, 2, \ldots, n$; $l = 1, 2, \ldots, k$) under the condition of this state of nature P_j. This is equivalent to conducting one complex experiment $\mathcal{E}$ with issues B_{ls} ($l = 1, 2, \ldots, k$; $s = 1, 2, \ldots, k$), where B_{ls} is marked the event, which consists of the fact that the first experiment gave B_l, and the second - B_s. The conditional probabilities of these issues according to product rule of the

probabilities of independent events will be: $P(B_{ls}/\Pi_j) = P(B_l/\Pi_j)\,P(B_s/\Pi_j)$. Thus problem is reduced to that previously examined, experiment will be only in no longer k of possible issues, but k^2.

So is the matter when repeated experimentation plan/glides previously. However, when speech occurs about conducting of a series of testings for refining the information about actual conditions in the situation in question, more advantageous not to assign the number of testings previously, but to solve after each testing - is worth to us carrying out following. It proves to be that this method in a series of the cases gives noticeable economy in the means, spent on experiment.

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10. METHOD OF NETWORK PLANNING.

§1. Problem of planning the complex of works.

During operations research in practice, frequently it is necessary to meet the problem of rational gliding/planning of complex, complicated works.

Examples of such works can be:

- building large industrial objective,

- the rearmament of army or separate branches of the armed forces.

- development/scanning the system of medical or preventive measures.

- the execution of composite scientific research theme with the

participation of a series of organizations, etc.

Characteristic for each such complex of works is the fact that it consists of a series of separate, elementary works or the "component/links" which are not simply implemented independently of each other, but they mutually cause each other, so that the execution of some works cannot be initiated earlier than are completed some others. Thus, for instance, during building of industrial enterprise the digging of foundation area cannot be initiated earlier than will be delivered and mounted the corresponding aggregate/units; piling foundation cannot be initiated earlier than will be delivered the necessary materials, for which, in turn, it is required the completion of building of entrance ways; for all stages of building, is necessary the presence of the corresponding technical specifications and records and, etc.

Gliding/planning for any such complex of works must be produced taking into account the following essential cell/elements:

- time, required for the execution of entire complex of works and its separate component/links.

- the cost/value of entire complex of works and its separate component/links.

- raw, energy and human resource/lifetimes.

The rational gliding/planning of the complex of works does require, in particular, answer/response to the following questions:

- How to distribute the available supplies and labor resource/lifetimes between the component/links of complex?

- In which torque/moments to begin and when to finish separate component/links?

- Which can arise obstructions to the timely completion of the complex of works and as remove them? and so forth.

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During gliding/planning of comparatively small by volume (quantity of component/links) complexes of works answer/response to such questions usually gives the leader, moreover without special mathematical calculations, it is simple on the basis of experiment and the common sense. However, when speech occurs about the very complex, expensive complexes of works, which consist of the large

number of component/links, by the complex form of those cause each other, such methods become not admitted. In these cases appears the need for the special calculations, which make it possible soundly to answer the placed above questions and a series of others.

One of the mathematical methods, widely used during the solution of this type of problems, is the method of network gliding/planning or as it frequently call, SPU (network management planning).

The method of network gliding/planning makes it possible to solve both direct and reverse problems of operations research. Direct problems do answer the question: which will be, if we do take this pattern of the organization of operation? Reverse do answer the question: how it is necessary to organize (to plan) operation so that it would possess, in some sense, maximum efficiency?

Inverse problems, as a rule, are much more complex than straight lines. In order to solve inverse problems, it is necessary first of all to learn to solve straight lines. It is logical, from such problems, we will begin.

Material for network gliding/planning is list or the enumeration of the works (component/links) of the complex, in which are shown not only works, but also their mutual conditionality (termination of

which works it is required for the beginning of the execution of each work). Let us call this list the structural table of the complex of works.

Table 1.1.

(1) № п/п	(2) Работа	(3) Опирается на работы
1	a_1	—
2	a_2	a_1, a_3
3	a_3	—
4	a_4	a_1, a_2, a_3
5	a_5	—
6	a_6	—
7	a_7	a_1, $a_{[illegible]}$, a_{10}
8	a_8	a_1, a_2
9	a_9	a_3, a_4, a_5
10	a_{10}	a_9
11	a_{11}	a_7, a_{12}
12	a_{12}	a_1, a_2
13	a_{13}	a_4, a_5, a_{10}
14	a_{14}	a_3, a_4, a_5
15	a_{15}	a_{10}

Key: (1). No.in sequence. (2). Work. (3). It is based on works.

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Let us agree to designate works a_1, a_2, ... In structural table for each work a_i must be shown that the executions of which works it requires, or as we will speak further, on which works it is based. An example of the structural table of works gives in Table 1.1.

In Table 1.1 last/latter column contains the enumeration of all works without completion of which this work cannot be initiated. Through-line in this graph means that this work can be initiated directly, right after making of a decision about conducting of the

complex of works.

The first operation which we will lead with structural table, is called ordering. With ordering to works is assigned certain new, more convenient numbering (each work can be based only on works with smaller sequence numbers).

Table 1.2.

(1) № п/п	(2) Работа	(3) Опирается на работы	(4) Ранг	(5) Обозначение в новой нумерации
1	a_1	—	1	b_1
2	a_2	a_1, a_3	2	b_5
3	a_3	—	1	b_2
4	a_4	a_1, a_2, a_3	3	b_6
5	a_5	—	1	b_3
6	a_6	—	1	b_4
7	a_7	a_1, a_4, a_{10}	6	b_{12}
8	a_8	a_1, a_2	3	b_7
9	a_9	a_3, a_6, a_5	4	b_9
10	a_{10}	a_9	5	b_{11}
11	a_{11}	a_7, a_{12}	7	b_{15}
12	a_{12}	a_1, a_9	3	b_8
13	a_{13}	a_4, a, a_{10}	6	b_{13}
14	a_{14}	a_3, a_4, a_5	4	b_{10}
15	a_{15}	a_{10}	6	b_{14}

Key: (1). No.in sequence. (2). Work. (3). It is based on works. (4). Rank. (5). Designation in new numbering.

Table 1.3.

(1) № п/п	(2) Работа	(3) Опирается на работы
1	b_1	—
2	b_2	—
3	b_3	—
4	b_4	—
5	b_5	b_1, b_2
6	b_6	b_1, b_5, b_2
7	b_7	b_1, b_5
8	b_8	b_1, b_5
9	b_9	b_2, b_6, b_3
10	b_{10}	b_9
11	b_{11}	b_9
12	b_{12}	b_1, b_6, b_{11}
13	b_{13}	b_6, b_3, b_{11}
14	b_{14}	b_{11}
15	b_{15}	b_{12}, b_{14}

Key: (1). No.in sequence. (2). Work. (3). It is based on works.

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For ordering of all works are subdivided into ranks. Work is called the work of the first rank, if for its beginning is required the executions of no other works. In Table 1.1, as we see that there are four works of the first rank: a_1, a_3, a_5 and a_6. Work is called the work of the second rank, if it is based on one or several works of the first rank. Work is called the work of the k th rank, if it is based on one or several works not above (k-1)-th rank, among which there is at least one work (k-1)-th rank.

After is produced the work assignment according to ranks, to works they are assigned new numbers, beginning with the works of the first rank, then the second, third and so forth. Within each rank of work, they are labeled in arbitrary order.

For an example let us produce the ordering of the works, placed in Table 1.1 (see Table 1.2). In the first two Table 1.2 gives: number and the designations of work in previous numbering, in two latter - a rank of work and its new designation in the regulated structural table.

After the ordering of works on ranks it is produced, it is possible to comprise the new, regulated table where the works are placed in the order of their new numbers (Table 1.3).

It is not difficult to see that in new, regulated structural Table 1.3 eacges of the works is based only on works with smaller reference numbers.

Subsequently, giving the structural tables of different complexes of works, we will be from the very beginning consider them regulated, and for works let us retain the initially undertaken designations: a_1, a_2 ....

2. Network schedule of the complex of works. Time/temporary network schedule.

Let us assume that to us is assigned the regulated structural table of the complex of the works, for example, with Table 2.1.

Communication/connections between the works, entering this complex, can be depicted graphically, in the form as of called network schedule (or graph/count).

Table 2.1.

(1) Работа	(2) Опирается на работу
a_1	—
a_2	—
a_3	—
a_4	a_1, a_2
a_5	a_2, a_3
a_6	a_4
a_7	a_5, a_6
a_8	a_3, a_5, a_6
a_9	a_7
a_{10}	a_5, a_8

Key: (1). Work. (2). It is based on work.

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This graph can be constructed differently. Most frequently represent the works, entering the complex, as rifleman/pointers, and the events, which consist of the execution of some works and possibility to begin new works - small circles or "units".

For an example let us represent in the form of network schedule structure Table 2.1 (Fig. 10.1). Units let us designate A_0, A_1, A_2, ..., works a_1, a_2, ...

The initial unit of entire complex of works let us designate A_0 and will understand hearth it the "beginning of works" or "making of

a decision about the beginning of works". From this unit proceed arrow/pointers a_1, a_2, a_3, the representing corresponding works and going respectively into units A_1, A_2, A_3. The combined execution of works a_1, a_2 we will represent as the further unit $A_{1,2}$, into which conduct broken pointers, which represent no works, but those indicating only logical communication/connection. From unit $A_{1,2}$, proceeds arrow/pointer a_4, that is based on works a_1, a_2; at the end of the arrow/pointer, stands unit A_4, which indicates the execution of work a_4, and so forth. The concluding unit A indicates the end of entire complex of works. On graph/count are lowered communication/connections, which logically escape/ensue from others; for example, work a_8 in Table 2.1 is based on works a_3, a_5, a_6; on graph/count it is shown that basing only on a_5, a_6, since work itself a_5 is based on a_3 and without its execution cannot be initiated.

As it was already said that there are different forms of the network schedules (for example, see [17]). In some by rifleman/pointers of graph/count are designated the works, and by units - the events, which consist of the execution of one or several works; in others - by units are designated the works, and by rifleman/pointers - logical communication/connections between them. Structural Table 2.1 it is possible to depict graphically, also, using this, second method (see Fig. 10.2). In this figure by heavy line are encircled small circles a_9, a_{10}, the representing

last/latter works of complex on which are based already no further works.

Each of the methods of the construction of network schedule has their advantages and their deficiency/lacks. An advantage of the second method, is that which into it is easy to introduce new, previously not indicated communication/connections which are discovered in the course of the execution of works. The advantage of the first, to view more complex, method is the fact that it can be comparatively simply adapted taking into consideration to the time of the execution of separate works and complex as a whole.

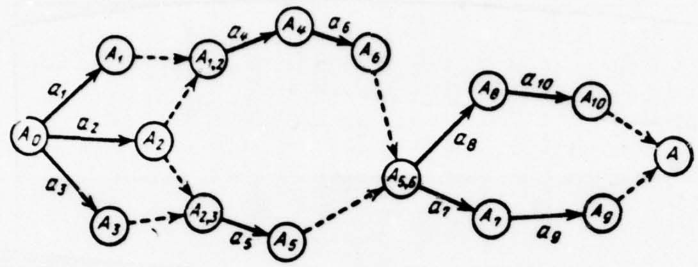

Fig. 10.1.

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Therefore we will use the first method.

Let us assume that to us is assigned the structural table of the complex of works, in which are written the times of the execution of each work: t_1, t_2, ... These values of the times of the execution of separate works we assume not random, but known previously. Will be obtained the table of form Table 2.2, in which structural communication/connections - the same as in Table 2.1 and on the graphs of Fig. 10.1, 10.2 (are reduced only "excess" communication/connection), but in right column are shown the times of the execution of separate works, expressed in any time units (watches, week, months). This table we will call

structural-time/temporary.

The time structure of the complex cf works, together with logical structure, can be depicted on one and the same graph which we will call time/temporary network schedule. We will construct as follows. Graph is oriented along the time axis 0t, on which on some scale are plot/deposited the times of the execution of works. As on of Fig. 10.1, of rifleman/pointers depicts works, but the length of each arrow/pointer is not arbitrary, but such, that its projection on the axis of abscissas 0t is equal to the time of the execution of this work. Logical communication/connections between works are designated as before by broken rifleman/pointers, who indicate no real work (sometimes them interpret as "fictitious" works).

Table 2.2.

(1) № п/п	(2) Работа a_i	(3) Опирается на работы	(4) Время t_i
1	a_1	—	10
2	a_2	—	5
3	a_3	—	15
4	a_4	a_1, a_2	18
5	a_5	a_2, a_3	19
6	a_6	a_4	18
7	a_7	a_5, a_6	8
8	a_8	a_5, a_6	25
9	a_9	a_7	30
10	a_{10}	a_8	8

Key: (1). No in sequence. (2). Work. (3). It is based on works. (4). Time.

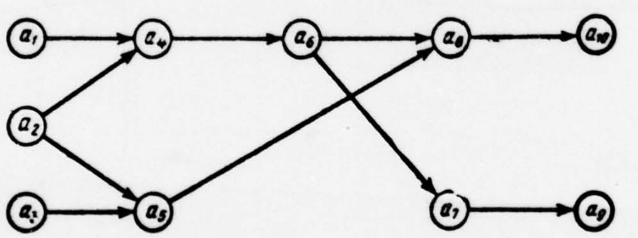

Fig. 10.2.

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The initial unit A_0 represents as before the beginning of the complex cf works; besides it, is introduced the final unit A, which represents the termination cf complex (this point of graph with the

greatest abscissa).

Time/temporary network schedule for the complex of works, assigned Table 2.2, shows on Fig. to 10.3. During the construction of time/temporary network schedule, the location of units on vertical line (along the axis of ordinates) is taken arbitrary, the abscissa of each unit is equal to the time of the termination of the corresponding work. The length of each arrow/pointer is counted from center to the center of small circle.

Let us observe how is constructed time/temporary network schedule in Fig. [redacted] 10.3. We begin it with unit A_0, placed in the beginning of coordinates. From this unit proceed three arrow/pointers: a_1, a_2, and a_3 whose projections on axis 0t are equal to the times of the execution of the corresponding works: $t_1 = 10$, $t_2 = 5$ and $t_3 = 15$. Work a_4 is based on works a_1 and a_2; of them work a_2 ends at torque/moment $t_2 = 5$, while work a_1 - at torque/moment $t_1 = 10$; that means work a_4 can be begun not earlier than at torque/moment $t_1 = 10$, when is finished a_1. Let us begin arrow/pointer a_4 from unit A_1, and unit A_2 is connected with A_1 by broken pointer. The projection of arrow/pointer a_4 is equal to t_4 = [redacted] 18, therefore, the abscissa of unit A_4, in which this arrow/pointer ends, must be $T_4 = t_1 + t_4 = 10 + 18 = 28$.

Arrow/pointer a_5, which is based on a_2 and a_3, must be begun in unit A_3, which has the greatest abscissa from $t_2 = 5$ and $t_3 = 15$; unit A_2 we combine with A_3 by broken pointer. Unit A_5, with which is finished arrow/pointer a_5, has abscissa $T_5 = t_3 + t_5 = 15 + 19 = 34$.

Arrow/pointer a_6 begins in unit A_4 and ends in unit A_6 with abscissa $T_6 = T_4 + t_6 = 28 + 18 = 46$. Arrow/pointer a_7, which is based on a_5, a_6, must begin from unit A_6, which has, in comparison with A_5, large abscissa; from A_5 in A_6, it is directed broken pointer. Arrow/pointer a_7 ends in unit A_7 with abscissa $T_7 = 46 + 8 = 54$. Arrow/pointer a_8 begins in the same unit A_6 with abscissa $T_6 = 46$ and ends in unit A_8 with abscissa $T_8 = 46 + 25 = 71$. Arrow/pointer a_9 with projection $t_9 = 30$ begins in unit A_7 and ends in unit A_9 with abscissa $T_9 = 54 + 30 = 84$. Arrow/pointer a_{10}, which is based on a_8, begins in unit A_8 and ends in unit A_{10} with abscissa $T_{10} = 71 + 8 = 79$.

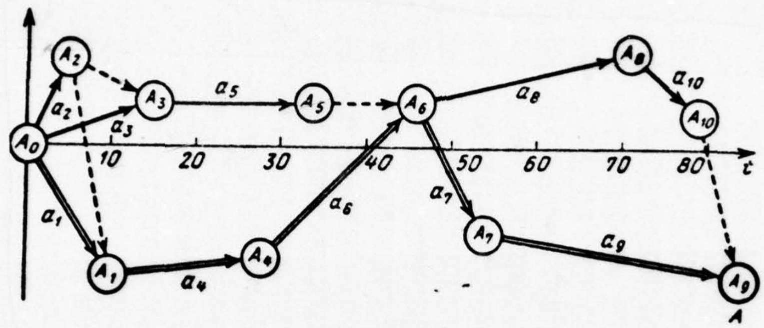

Fig. 10.3.

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Since work a_9 is completed by the latter, then unit A_9 = A indicates the termination of entire complex of works; let us note this unit by greasy/fatty small circle and is connected with it by broken pointer unit A_{10} - termination of the previous work a_{10}, on which, besides the end of the works, nothing is based.

Thus, the time/temporary network schedule of the complex of the works is constructed.

Time T = 84 from initial unit A_0 to concluding A = A_9 represents by itself the minimum time in which can be completed the complex of

works.

Let us focus attention on the following fact: time T represents by itself the sum of the times of the performance not of all works, but only some of them:

$$T = t_1 + t_4 + t_6 + t_7 + t_9 = 10 + 18 + 18 + 8 + 30 = 84.$$

Works a_1, a_4, a_6, a_7, a_9 from durations of which is comprised time T, are called critical works, and the chain/network of the corresponding to them arrow/pointers on network schedule - critically. Figures 10.3 critical way shows double rifleman/pointers.

The special feature/peculiarity of critical works consists of following: so that would be observed the minimum time of the execution of complex, each of them must begin accurately at the torque/moment when was finished the latter from the works on which it is based, and to be continued on and what is more the time which by it is assigned according to the plan; least retardation in execution by each of the critical works leads to corresponding delay of the fulfilment of plan as a whole. Thus, critical way on network schedule - this is the set of the most vulnerable, "weak places" of plan/layout, which must be placed in time/temporary plan/layout with the greatest clearness. As concerns remaining, "noncritical" works (in our case this a_2, a_3, a_5, a_8, a_{10}), then with them matter is not

so badly/poorly: each of these works has known time/temporary reserves and can be finished with a certain delay without this would be reflecting in the period of the execution of complex as a whole.

The reserves, which correspond to noncritical works, easily can be determined by network schedule.

Let us name "noncritical arc" the set of noncritical works and units, which begins and which ends on critical way (taking into account and broken pointers). In Fig. 10.3 are four noncritical arcs:

$$\begin{aligned}&A_0 - a_2 - A_2 - A_1,\\&A_0 - a_3 - A_3 - a_5 - A_5 - A_6,\\&A_0 - a_2 - A_2 - A_3 - a_5 - A_5 - A_6,\\&A_6 - a_8 - A_8 - a_{10} - A_{10} - A_9.\end{aligned}$$

On the first of these arcs, lie/rests one noncritical work

a_2; to the second - two noncritical works a_3, a_5; on the third - two noncritical works a_2, a_5; on the fourth - two noncritical works a_8, a_{10}.

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To each noncritical arc corresponds the specific time/temporary reserve which can be by arbitrary form distributed between the noncritical works, which lie on this arc. This reserve is equal to

the difference between the sum of the times of the critical works, which lie on the critical way, closing arc, and noncritical, that lie cn arc itself.

For example, on arc A_0-a_2-A_2-A_1 lie/rests only one noncritical work a_2; on its closing segment of the critical way A_0-a_1-A_1 - one critical work a_1; the reserve of time, which is necessary to work a_2, is equal to $R_2 = t_1 - t_2 =$ [redacted] $10-5 = 5$. Consequently, the execution of work a_2 can be, without damage for a common/general/total period, delayed per 5 time units.

On the second noncritical arc A_0-a_3-A_3-a_3-A_5-A_6, lie/rest two noncritical works a_3, a_5; on the closing section of critical way - three critical works a_1, a_4, a_6. That means that the general reserve cf time, which is necessary to works a_3, a_5, is equal to:

$$R_{3,5} = t_1 + t_4 + t_6 - (t_3 + t_5) = 10 + 18 + 18 - (15 + 19) = 12.$$

By [redacted] it is possible any form to distribute between works a_3, a_5.

On the third noncritical arc the reserve is equal to:

$$R_{2,5} = t_1 + t_4 + t_6 - (t_2 + t_5) = 10 + 18 + 18 - (5 + 19) = 22.$$

Since to us already it is known that with work a_2 we can retard nct more than per 5 time units, and with work a_5 - is not more than to 12, the requiræment, so that the sum of delays would be not more

than 22, it tells us nothing new.

On the fourth noncritical arc there is the reserve

$$R_{8.10} = t_7 + t_9 - (t_8 + t_{10}) = 8 + 30 - (25 + 8) = 5,$$

which can be arbitrarily distributed between works a_8, a_{10}.

The knowledge of critical way on network schedule is useful in two relations: first, it makes it possible to isolate from entire complex of works the set of those most" threatened", continuous to control them, and, in the case of the necessity, to boost their execution; in the second place, it makes it possible in principle to accelerate the execution of the complex of works because of from "harmless" retarding/deceleration/delay to move the part of the forces and means to more important, more critical works.

Let us note that on network schedule generally there can be not one critical way, but two or it is more; logically the sum of the times of critical works for each of them must be one and the same.

Example. To construct timeline for the complex of works, given into a structural-time/temporary Table 2.3. To find on it critical way (or ways, if them several) and to determine the reserves of time on noncritical arcs.

Solution. The network schedule of complex is given in Fig. 10.4. Critical way is designated by double rifleman/pointers and consists of the works:

$$a_1,\ a_5,\ a_7,\ a_9,\ a_{12}.$$

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However, it is possible to construct another critical way, which consists, for example, of the works

$$a_4,\ a_6,\ a_7,\ a_{10},\ a_{12};$$

in this case time T of the termination of the complex of works on both critical ways it will be one and the same:

$$\begin{aligned} T &= t_1 + t_5 + t_7 + t_9 + t_{12} = 20 + 10 + 20 + 10 + 20 = 80, \\ T &= t_4 + t_6 + t_7 + t_{10} + t_{12} = 20 + 10 + 20 + 10 + 20 = 80. \end{aligned}$$

Besides these two on graph (Fig. 10.4) can be discovered still some critical ways; we let for reader to find them independently.

It is isolated on network schedule (Fig. 10.4) four noncritical arcs:

$$\begin{aligned} &A_0 - a_2 - A_2 - A_1, \\ &A_0 - a_3 - A_3 - A_1, \\ &A_5 - a_8 - A_8 - A_7, \\ &A_9 - a_{11} - A_{11} - A_{12}. \end{aligned}$$

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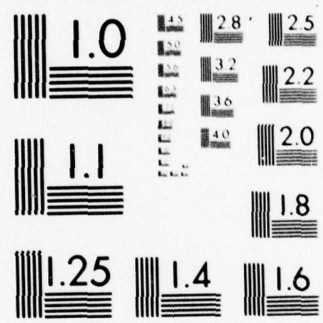
1.0
2.8
2.5
3.2
2.2
3.6
4.0
2.0
1.1
1.8
1.25
1.4
1.6

Table 2.3.

(1) № п/п	(2) Работа a_i	(3) Опирается на работы	(4) Время t_i
1	a_1	—	20
2	a_2	—	15
3	a_3	—	10
4	a_4	—	20
5	a_5	a_1, a_2, a_3	10
6	a_6	a_3, a_4	10
7	a_7	a_5, a_6	20
8	a_8	a_5, a_6	15
9	a_9	a_7, a_8	10
10	a_{10}	a_7, a_8	10
11	a_{11}	a_9, a_{10}	10
12	a_{12}	a_9, a_{10}	20

Key: (1). No. in sequence. (2). Work. (3). It is based on works. (4). Time.

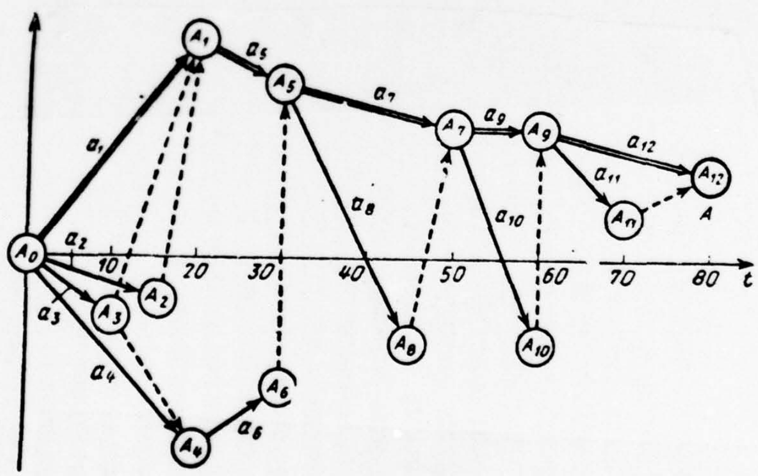

Fig. 10.4.

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For each of these arcs, there is from one noncritical work: a_2, a_3, a_8 and a_{11}. The reserves of time for them are equal respectively:

$R_2 = t_1 - t_2 = 5$; $R_3 = t_1 - t_3 = 10$; $R_8 = t_2 - t_8 = 5$; $R_{11} = t_{12} - t_{11} = 10$.

The noncritical arc $A_0 - a_3 - A_3 - A_4$ under our conditions does not give the new data on the reserves of time.

3. Formal recording (algorithm) of the problem of network gliding/planning.

The described above graphic method of construction and analysis of job schedule is suitable only in the case when the planned/glide complex is not too complex (in a quantity of works and logical communication/connections). In practice frequently are encountered the complexes of works, which consist of the enormous number of component/links (on the order of thousands and more), by the complex form of those basing on each other. It is logical that in such cases

the drawing of network schedule by hand - heavy and thankless occupation, since the major advantage of network schedule - its clarity - in this case is lost. For analysis and improvement of job schedule in such cases, they usually draw FTsVM.

So that the machine would be capable of producing the appropriate actions, it is necessary to completely formalize the procedure of the construction of network schedule, to express it in the form of the precise sequence of actions or algorithm.

Let us describe one of the possible algorithms which can be used for this purpose. First of all, is implemented the ordering of the structural table (see §2), for which a work they are divided into ranks, according to the sign/criterion of number and ranks of the works on which they are based. This - a comparatively simple problem, and on it we be stopped will not be. Let us assume that the ordering of the complex of the works is carried out, and structural- table depicts, for example, in the form Table 3.1.

Let us register in the form of mathematical formulas the communicating system, reflected into a structural-time/temporary to the table of complex.

Table 3.1.

(1) № п/п	(2) Работа a_i	(3) Опирается на работы	(4) Время t_i
1	a_1	—	t_1
2	a_2	—	t_2
3	a_3	—	t_3
4	a_4	a_1, a_2	t_4
5	a_5	a_1, a_3, a_4	t_5
6	a_6	a_2, a_3	t_6
7	a_7	a_6	t_7
8	a_8	a_4, a_5	t_8
9	a_9	a_4, a_5, a_6	t_9
10	a_{10}	a_8, a_9	t_{10}

Key: (1). in sequence. (2). Work. (3). It is based on works. (4). Time.

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For this, let us designate τ_i - smallest possible period of the beginning of work a_i (time is counted off from the beginning of process), and T_i - the smallest possible period of its termination. it is obvious

$$T_i = \tau_i + t_i, \tag{3.1}$$

where t_i - time of the execution of work a_i.

Using these designations, it is possible to register by formulas all logical communication/connections between the works of complex. It is real/actual, let, for example, the work a_i be based on works a_j, a_l, a_k. Then the work a_i cannot be begun earlier than will end that

cf the works a_j, a_l, a_k, which it ends more late than all. This communication/connection we will register in the form of the formula

$$\tau_i = \max \{T_j, T_l, T_k\}. \qquad (3.2)$$

Applying such formulas to all works of complex on turn, we will find all torque/moments of the termination of works T_i and, after all, the minimum period of the termination of entire complex of works T.

Let us demonstrate how this is made on material Table 3.1. Let us compute values τ_i and T_i for all works of complex.

For the works of the first rank a_1, a_2, a_3, we have:

$$\tau_1 = 0; \quad T_1 = t_1;$$
$$\tau_2 = 0; \quad T_2 = t_2;$$
$$\tau_3 = 0; \quad T_3 = t_3.$$

Work a_4 is based on works a_1, a_2; it can be begun at torque/moment τ_4, when is finished most late ending from works a_1, a_2:

$$\tau_4 = \max \{T_1, T_2\}. \qquad (3.3)$$

Torque/moment of the termination of work a_4:

$$T_4 = \tau_4 + t_4. \qquad (3.4)$$

Analogously, for works a_5, a_6 and so forth:

$$\left.\begin{aligned}
\tau_5 &= \max\{T_1, T_3, T_4\};\\
T_5 &= \tau_5 + t_5;\\
\tau_6 &= \max\{T_3, T_2\};\\
T_6 &= \tau_6 + t_6;\\
\tau_7 &= \max\{T_4\} = T_4;\\
T_7 &= \tau_7 + t_7;\\
\tau_8 &= \max\{T_4, T_5\};\\
T_8 &= \tau_8 + t_8;\\
\tau_9 &= \max\{T_4, T_5, T_6\};\\
T_9 &= \tau_9 + t_9;\\
\tau_{10} &= \max\{T_8, T_9\};\\
T_{10} &= \tau_{10} + t_{10}.
\end{aligned}\right\} \quad (3.5)$$

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Thus, are found the torque/moments of beginning τ_i and ending T_i of all works of complex. It is logical that the time of the termination of entire complex is equal to maximum of all times of the termination:

$$T = \max\{T_1, T_2, \ldots, T_{10}\}. \quad (3.6)$$

In order to find the critical works (but therefore, and critical way), it is necessary to do following; to first of all find work a_i, for which the time of termination $T_i = T$ is maximal; this work, of course, will be critical. Then to find among formulas (3.5) that by which is determined the torque/moment of the beginning of this work τ_i. Value τ_i is represented in the form of the maximum of some torque/moments $T_j, T_k, T_l, \ldots$; it is necessary to find that from them, on which is reached the maximum (if such torque/moments several, to take

any of them). That work a_m, at which is reached this maximum, will be the second from end work on critical way. Further in exactly the same manner is determined the third and so forth of work on critical way. Thus, critical there will be the work with the latest period of termination and all works, during the period of termination of which is reached the maximum in the expression, which determines the period of the beginning of next critical work. It is certain, maximum in any of formulas (3.5) can be reached not on one, but on several works; respectively at each step/pitch we can obtain not one, but several critical ways.

Let us demonstrate the algorithm of the construction of critical way on the material of the same structural-time/temporary Table 3.1. For this, obviously, it is necessary to assign in this table the concrete/specific/actual values of times t_i (Table 3.2).

Table 3.2.

(1) № п/п	(2) Работа a_i	(3) Опирается на работы	(4) Время t_i
1	a_1	—	15
2	a_2	—	12
3	a_3	—	20
4	a_4	a_1, a_2	10
5	a_5	a_1, a_3, a_4	15
6	a_6	a_2, a_3	18
7	a_7	a_4	40
8	a_8	a_4, a_5	8
9	a_9	a_4, a_5, a_6	23
10	a_{10}	a_8, a_9	11

Key: (1). in sequence. (2). Work. (3). It is based on works. (4). Time.

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We have:

$$T_1 = 15; \quad T_2 = 12; \quad T_3 = 20;$$

$$\tau_4 = \max\{T_1, T_2\} = \max\{15, 12\} = 15. \tag{3.7}$$

$$T_4 = \tau_4 + t_4 = 15 + 10 = 25;$$

$$\tau_5 = \max\{T_1, T_3, T_4\} = \max\{15, 20, 25\} = 25. \tag{3.8}$$

$$T_5 = \tau_5 + t_5 = 25 + 15 = 40;$$

$$\tau_6 = \max\{T_2, T_3\} = \max\{12, 20\} = 20. \tag{3.9}$$

$$T_6 = \tau_6 + t_6 = 20 + 18 = 38;$$

$$\tau_7 = \max\{T_4\} = T_4 = 25. \tag{3.10}$$

$$T_7 = \tau_7 + t_7 = 25 + 40 = 65;$$

$$\tau_8 = \max\{T_4, T_5\} = \max\{25, 40\} = 40. \tag{3.11}$$

$$T_8 = \tau_8 + t_8 = 40 + 8 = 48;$$

$$\tau_9 = \max\{T_4, T_5, T_6\} = \max\{25, 40, 38\} = 40. \tag{3.12}$$

$$T_9 = \tau_9 + t_9 = 40 + 23 = 63;$$

$$\tau_{10} = \max\{T_8, T_9\} = \max\{48, 63\} = 63 \tag{3.13}$$

$$T_{10} = \tau_{10} + t_{10} = 63 + 11 = 74.$$

The time of the execution of entire complex of works is maximum of all times T_i, i.e. $T_{10} = 74$:

$$T = \max\{T_1, T_2, \ldots T_{10}\} = 74. \tag{3.14}$$

Let us find critical works, beginning with the latter. Since maximum in formula (3.14) is reached for T_{10}, this work a_{10} is critical. It is based on works a_8 and a_9. Which of them is critical? It is obvious, a_9, since maximum in formula (3.13) falls on T_9. We pass to formula (3.12) - in it maximum falls on T_5, which means, that work a_5 - is critical. Further, analogously, examine/scanning maximums in formulas (3.11)-(3.7), we find one after another all critical works. Enumerated in natural order, they will be: a_1, a_3, a_4, a_5, a_9, a_{10}.

Thus, critical way is found by purely formal method, without the construction of network schedule. In Table 3.2 are emphasized all critical works.

Completely analogous formal can be determined noncritical arcs, and the corresponding to them reserves.

4. Optimization of the plan/layout of the complex of works.

We already spoke about the fact that the network schedule (or its substituting formal algorithm of the analysis of the complex of works) can be used for an improvement (optimization) in the plan/layout.

This improvement can be produced for different target/purposes. For example, it can seem that the total time of carrying out the complex of works T us is not tripled; arises the question concerning how it is necessary to boost works, so that the total time would not exceed the assigned period T_0. It is obvious, for this has the sense to boost the precisely critical works a reduction in duration of which directly will pronounce on time T. However, in this case, it can seem that the critical way will be changed, and the weakest places on time will render/show some other works. It is logical to assume that the boosting of works is given not for free, but it requires the insertion of some means. Does appear the typical problem of operations research: which further resources and into which works one should put so that the common/general/total period of the execution of the complex of works would be not more than assigned magnitude T_0, but the expenditure of further resources was minimum?

Another problem of optimization is related to the redistribution of the already available resources between separate works. Above we ascertained that all works, except critical, have some time/temporary reserves.

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In certain cases it proves to be possible, after moving forces and

resources from the noncritical sections of plan/layout to critical, to attain the decrease of the total time of the fulfilment of plan. Again does appear the typical problem of operations research: which forces and resources must be moved from some works to others so that the time of the execution of the complex of works would become minimum?

Finally, is possible one additional formulation of the problem of the optimization of plan/layout. After the construction of network schedule to us it became it is known that the minimum time of the execution of entire complex of works is placed in the assigned period with the surplus:

$$T < T_0,$$

i.e. of us exists the known reserve of time, by which we right to be ordered, having somewhat stretched works (but, it goes without saying, so as not to leave in the assigned period T_0). After stretching works, we can economize some resources. Does arise the question: to which limits it is possible to increase the times of the execution of works and which works so that the obtained from this economy of resources would be maximum? In this setting can be placed the problem of optimization not necessarily of entire plan/layout, and it can be, the separate noncritical arcs, on which are revealed time/temporary reserves.

Let us give the setting each of these problems optimization in formula recording. For simplicity let us assume that critical way - one (if this not then, obviously, it is always possible, introducing into the times of the execution of the works of ε-change, to do critical path only, similarly, as we entered the degenerate problems of linear programming).

Problem 1. Complex consists of works $a_1, a_2, \ldots, a_n$ at times of execution $t_1, t_2, \ldots, t_n$; is known critical way, moreover the time of the execution of complex is equal

$$T=\sum_{(\text{кр})} t_i > T_0, \qquad (4.1)$$

where the addition extends only to critical works. The assigned period of the execution of the complex of works is equal to T_0.

It is known that the insertion of the specific sum x of further resources of work a_i shortens the time of execution with t_i to $t_i' = f_i(x) < t_i$.

It asks itself, which further resources $x_1, x_2, \ldots, x_n$ should put in each of the works, in order to:

- period of the execution of complex was not higher than given one T_0,

- a sum of the inserted resources it reached the minimum.

Thus, we need to determine the nonnegative values of variables $x_1, x_2, \ldots, x_n$ (further insertions) so that would be implemented the condition

$$T' = \sum_{(\kappa p)} f_i(x_i) \leqslant T_0, \qquad (4.2)$$

where the addition extends from all critical works of new critical way (obtained after the redistribution of resources and change in the times), and so that in this case the common/general/total sum supplement body insertions it would be minimum:

$$x = \sum_{i=1}^{n} x_i = \min. \qquad (4.3)$$

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Stated problem calls to mind the problem of linear programming, because in it with some limitation-inequalities it is required to minimize linear function (4.3) of the cell/elements of solution. However, in the general case the entering limitations (4.2) functions $f_i(x_i)$ are nonlinear, since the insertion of some resources into work a_i does not compulsorily cause the linear decrease of the time, spent on this work. Therefore in general form stated problem is related to the class of the problems of the nonlinear programming which much more complex than linear problems and methods of solution of which are not worked out. Such nonlinear problems we here be occupied will not be, referring those interesting to special

management/manuals ([2, 28]). However, if we are bounded to the comparatively small changes in the plan/layout (by such, during which the time of the execution of works approximately linearly it depends on the inserted further resources, and critical way it does not vary), stated problem it becomes the problem of linear programming it can be solved by the already known to us methods (see Chapter 2).

Example 1. There is a complex of works a_1, a_2, ..., a_8 whose parameters are given into a structural-time/temporary Table 4.1

The network schedule of works gives in Fig. 10.5. The completion of work - unit $A = A_8$; critical way consists of works a_1, a_4, a_8. Time of the termination of the complex:

$$T = t_1 + t_4 + t_8 = 50.$$

This time it must be **decreased to T_0 = 40**; for this by us it will be required to boost certain critical works. It is known that into work a_i it is possible to put resources x_i in size/dimension not more than c_i. in this case the time of the execution of work is reduced according to the linear dependence:

$$x_i < c_i, \quad t_i' = t_i(1 - b_i x). \tag{4.4}$$

Table 4.1.

(1) № п/п	(2) Работа a_i	(3) Опирается на работы	(4) Время t_i
1	a_1	—	20
2	a_2	—	10
3	a_3	—	8
4	a_4	a_1, a_2	20
5	a_5	a_1, a_2, a_3	10
6	a_6	a_1, a_2, a_3	5
7	a_7	a_6	5
8	a_8	a_4, a_5, a_7	10

Key: (1). in sequence. (2). Work. (3). It is based on works. (4). Time.

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For critical works a_1, a_4, a_8, the parameters b_i, c_i are equal to

$$\left.\begin{aligned} b_1 &= 0{,}2,\ c_1 = 2,\\ b_4 &= 0{,}3,\ c_4 = 2,\\ b_8 &= 0{,}1,\ c_8 = 5. \end{aligned}\right\} \qquad (4.5)$$

It is required to determine insertions x_1, x_4, x_8 so that the period of the execution of complex would be not more than $T_0 = 40$, but to the sum of insertions it reached the minimum:

$$x_1 + x_4 + x_8 = \min.$$

Solution. Conditions (4.4) and (4.5) give:

$$2 - x_1 > 0;\ 2 - x_4 > 0;\ 5 - x_8 > 0. \qquad (4.6)$$

The new period of the execution of works (when critical way it will not be changed)

$$T' = t_1' + t_4' + t_8' = t_1(1 - 0{,}2x_1) + t_4(1 - 0{,}3x_4) + t_8(1 - 0{,}1x_8) =$$
$$= 20(1 - 0{,}2x_1) + 20(1 - 0{,}3x_4) + 10(1 - 0{,}1x_8) = 50 - 4x_1 - 6x_4 - x_8.$$

This value must not exceed $T_0 = 40$:

$$50 - 4x_1 - 6x_4 - x_8 < 40,$$

whence

$$4x_1+6x_4+x_8 > 10. \qquad (4.7)$$

Let us assign the mission of the linear programming: to find the nonnegative values of the variables x_1, x_1, x_8, that satisfy condition-inequalities (4.6), (4.7) and rotating in the minimum the linear function

$$L=x_1+x_4+x_8. \qquad (4.8)$$

We solve problem according to the general rules of the simplex method (see Chapter 2).

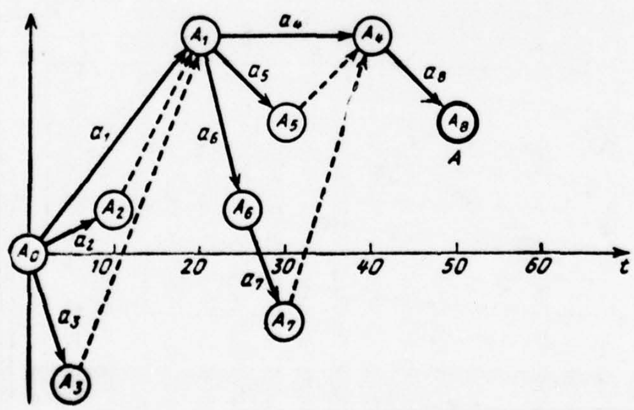

Fig. 10.5

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Table 4.2.

	Свободный член (1)	x_1	x_4	x_8
L	0	-1	-1	-1
y_1	2	1	0	0
y_2	2	0	1	0
y_3	5	0	0	1
y_4	-10	-4	-6	-1

Key: (1). Absolute term.

Table 4.3.

		Свободный член (1)	x_1	x_4 $y_4 \leftrightarrow$	x_8
	y_1	2 0	1 0	0 $\boxed{0}$	0 0
	y_2	2 $-\frac{5}{3}$	0 $-\frac{2}{3}$	1 $\boxed{\frac{1}{6}}$	0 $-\frac{1}{6}$
	y_3	5 0	0 0	0 $\boxed{0}$	1 0
$x_4 \leftrightarrow$	y_4	$\boxed{-10}$ $\frac{5}{3}$	$\boxed{-4}$ $\frac{2}{3}$	-6 $-\frac{1}{6}$	$\boxed{-1}$ $\frac{1}{6}$

Key: (1). Absolute term.

Table 4.4

	Свободный член (1)	x_1	y_4	x_8
L	$\frac{5}{3}$	$-\frac{1}{3}$	$-\frac{1}{6}$	$-\frac{5}{6}$
y_1	2	1	0	0
y_2	$\frac{1}{3}$	$-\frac{2}{3}$	$\frac{1}{6}$	$-\frac{1}{6}$
y_3	5	0	0	1
x_4	$\frac{5}{3}$	$\frac{2}{3}$	$-\frac{1}{6}$	$\frac{1}{6}$

Key: (1). Absolute term.

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By the introduction of the further variables y_1, y_2, y_3, y_4 condition-inequality (4.6), (4.7) they are converted into the equalities:

$$\left.\begin{aligned} y_1 &= 2 - x_1, \\ y_2 &= 2 - x_4, \\ y_3 &= 5 - x_8, \\ y_4 &= 4x_1 + 6x_4 + x_8 - 10. \end{aligned}\right\}$$

we comprise simplex-table (Table 4.2). Set/assuming unrestricted variables equal to zero: $x_1 = x_4 = x_8 = 0$, will obtain the inadmissible solution in which $y_4 = -10$; therefore the supporting/reference solution of OZLP is required still to find. We enter according to the general rule of the determination of the supporting/reference solution of OZLP (see §7 Chapters 2). reject/throwing temporarily row L (during the determination of supporting/reference solution it it is not necessary) and choosing as

the solving cell/element cell/element -6 in last/latter row, we will obtain a simplex-table (Table 4.3).

Continuing actions, we come to the supporting/reference solution, registered in Table 4.4.

In Table 4.4 all absolute terms are already positive, which means, that supporting/reference solution is found. In row L table 4.4 is placed (in standard form) the linear function L, expressed through the new unrestricted variables x_1, y_4, x_8:

$$L = x_1 + x_4 + x_8 = x_1 + {}^5/_3 - {}^2/_3 x_1 + {}^1/_6 y_4 - {}^1/_6 x_8 + x_8 =$$
$$= {}^5/_3 + {}^1/_3 x_1 + {}^1/_6 y_4 + {}^5/_6 x_8 = {}^5/_3 - (-{}^1/_3 x_1 - {}^1/_6 y_4 - {}^5/_6 x_8).$$

All coefficients in upper row Table 4.4 are negative; consequently, an increase each z of unrestricted variables can only increase function L. That means that optimum solution is found:

$$x_1 = y_4 = x_8 = 0,$$
$$y_1 = 2, \; y_2 = 1/3, \quad y_3 = 5, \quad x_4 = 5/3.$$

At these values the alternating/variable sum of insertions it reaches the minimum:

$$L = L_{min} = 5/3.$$

Thus, optimum solution by the insertion of resources is following: to put sum $x_4 = 5/3$ of work a_4; into works a_1 and a_8 not to pack resources. In this case, period T' of the execution of works will be:

$$T' = t_1 + t_4' + t_8 = 20 + 20\,(1 - 0{,}3 \cdot 5/3) + 10 = 20 + 10 + 10 = 40 = T_0.$$

It is checked, will be preserved during this solution critical way?

Figures 10.5 shows that reduction t_4 from 20 to 10 does not still vary critical way, but it is located already on the very boundary of that reduction, during which critical way varies.

Arises the natural question: a what is to be done, if with the insertion of resources into some works critical way changes. It appears in this case the problem of optimization also can be reduced to the problem of linear programming, but to other - already more complex, with the large number of variables. Let us show how this is made, based on the same example, but in literal form, without leading to numerical results.

As variables let us introduce resources $x_1, \ldots, x_8$, packed into works $a_1, \ldots, a_8$; torque/moments $\tau_4, \ldots, \tau_8$ the beginning of works $a_4, \ldots, a_8$ (torque/moments τ_1, τ_2, τ_3 are equal to 0) and torque/moments T_1, T_2, ..., T_8 of end of all works.

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Structural table gives to us the following limitation-inequalities:

$$\tau_4 \geqslant T_1, \quad \tau_4 \geqslant T_3;$$
$$\tau_5 \geqslant T_1, \quad \tau_5 \geqslant T_2, \quad \tau_5 \geqslant T_3;$$
$$\tau_6 \geqslant T_1, \quad \tau_6 \geqslant T_2, \quad \tau_6 \geqslant T_3;$$
$$\tau_7 \geqslant T_6;$$
$$\tau_8 \geqslant T_4; \quad \tau_8 \geqslant T_5, \quad \tau_8 \geqslant T_7.$$

The conditions of the dependence of the time of the execution of work on the inserted resources will give to us limitation-equalities:

$$T_i = t_i - t_i b_i x_i + \tau_i \quad (i = 1, \ldots, 8)$$

(recall that here t_i and b_i - constants). Further, are retained condition-inequalities

$$x_i < c_i, \quad (i = 1, \ldots, 8).$$

Finally, the condition of the execution of entire complex of works into period will become limitation-inequalities

$$T_1 \leqslant T_0, \ldots, T_8 \leqslant T_0,$$

from which, by special feature/peculiarity strength of this concrete/specific/actual structural table, it is possible to leave only the last/latter: $T_8 \leqslant T_0$.

Under all these conditions it is necessary to minimize the linear function

$$L = x_1 + \ldots + x_8.$$

Thus, problem was reduced to the problem of the linear programming from 21 of variable, with 8 limitation-inequalities even 21 by a limitation-inequality; by the introduction of further variables it it is possible to reduce to OZLP with 42 variables and 29 limitation-equalities. It is certain problem with family alternating/variable and four limitation-equalities many times it is simpler; so that in such cases it is reasonable first to check that will not be preserved critical way previous, as this was at numerical

values $t_i, b_i\ c_i$, examined in example 1.

Problem 2. There is a set of the works: $a_1, a_2, \ldots, a_n$ at times of execution $t_1, t_2, \ldots, t_n$. The time of the execution of the complex of works is expressed by the formula

$$T = \sum_{(\text{кр})} t_i. \tag{4.9}$$

On noncritical works there are some time reserves using these reserves, i.e., moving some resources from noncritical works to critical, it is possible to decrease the times of the execution of critical works and thereby the time of execution of entire complex.

There is certain constant/invariable supply of the movable resources B which is distributed between works $a_1, a_2, \ldots, a_n$ so that to each work corresponds the quantity of movable resources, equal to respectively

$$b_1, b_2, \ldots, b_n;\quad \sum_{i=1}^{n} b_i = B.$$

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It is known that the quantity of resources x > 0, removed from work a_i, increases the time of its execution with t_i to

$$t_i' = f_i(x) > t_i,$$

a quantity of resources x, inserted additionally into work a_i, reduces the time of its execution to

$$t_i'' = \varphi_i(x) < t_i.$$

It does ask itself: how it is necessary to redistribute the available movable resources B between works so that the period of the execution of complex would be minimum?

Let us show how can be solved a similar problem.

Let us designate x_i - quantity of movable resources, thrown to work a_i (x_i is negative, if from work a_i is remove/taken some quantity of resources).

Values x_i must satisfy the limitations:

$$x_i \geqslant -b_i \quad (i=1, \ldots, n). \tag{4.10}$$

It is logical that the sum of the resources, removed from some works, must be equal to the sum of the resources, supplemented to other works, so that

$$x_1 + \ldots + x_n = 0. \tag{4.11}$$

After the transfer of resources for those works, to which they are moved, new times will be equal to

$$t_i' = \varphi_i(x_i), \tag{4.12}$$

for the same works from which the resources are remove/taken:

$$t_j'' = f_j(|x_j|). \tag{4.13}$$

The common/general/total period of the execution of the complex of works will be:

$$T' = \sum_{\substack{(\text{кр}) \\ i}} \varphi_i(x_i) + \sum_{\substack{(\text{кр}) \\ j}} f_j(|x_j|). \tag{4.14}$$

where the first sum extends to all works to which are transferred the resources, if they enter in critical way; the second - to all works from which are transferred the resources, if they enter in critical way.

It is logical it seems that to consider that transfer of resources makes sense only from noncritical works to critical; however not above to forget, that in this case the noncritical works can pass into critical, but vice versa; therefore in formula (4.14) in the general case they are present both sums.

Thus, before us stands the problem: to find such values of variables x_i (i = 1, ..., n), so that were implemented limitations (4.10), and function (4.14) was converted into the minimum.

Problem represents by itself the problem of nonlinear programming even in the case when function f_i and φ_i (that with certain tension it is possible to allow) are linear. Substantially nonlinear in function (4.14) is the fact that values i, j - numbers of the works to which extends the sum (i.e. critical works), themselves depend on x_i.

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As has already been spoken, methods of the solution of the problems of nonlinear programming worked out; however sometimes it is possible to solve a similar kind of problem, using comparatively simple methods. In the following example we will consider solution of one of such problems.

Example 2. The complex of works a_1, a_2, a_3 is assigned structural-time/temporary Table 4.5. Critical are here works a_1, a_3; the time of the execution of complex T = 30. noncritical is work a_2. On it there is a supply of the movable resources $b_2 = 1$. The supplies of movable resources on remaining two works are absent.

It is known that the transfer of resources x from work a_2 increases the time of its execution to

$$t''_2 = \frac{10}{1-0,1x}.$$

The supply of resources x, moved by work a_1, reduces the time of its execution to

$$t_1' = \frac{20}{1+x}.$$

The supply of resources x, moved by work a_3, reduces the time of its execution to

$$t_3' = \frac{10}{1+4x}.$$

It is required to define as to optimally transfer free resources from work a_2 to works a_1 and a_3 so that the period of the execution of complex would be maximum.

Solution. Let us designate the quantities of resources, moved from work a_2 respectively on a_1 and $_3$, through x_1 and x_3. It is required to find such nonnegative values x_1 and x_3 so that would be implemented **the conditions:**

$$x_1+x_3<1, \qquad (4.15)$$

$$(t_1')_{\text{кр}}+(t_2'')_{\text{кр}}+(t_3')_{\text{кр}}=\min, \qquad (4.1\ )$$

where the index "kr" means that the corresponding term enters in sum only if it it belongs to critical way.

Let us look, under which conditions work a_2 will enter into critical way. This will occur, if the time of its execution becomes more than the time of the execution of work a_1:

$$t_2''>t_1'.$$

Table 4.5

(1) № п/п	(2) Работ: a_i	(3) Опирается на работы	(4) Время t_i
1	a_1	—	20
2	a_2	—	10
3	a_3	a_1, a_2	10

Key: (1). in sequence. (2). Work. (3). It is based on works. (4). Time.

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The "jump/migration" of work a_2 to critical way occurs, when is realized the equality:

$$t_2' = t_1'$$

or (if from work a_2 are removed all resources)

$$\frac{10}{1-0,1} = \frac{20}{1+x_1},$$

which is realized with x_1 = 0.8. That means that work a_1 and a_3. Let us assume that this so and in formula (4.16) the second term will be equal to zero, but two others t'_1 and t'_3:

$$T' = t_1' + t_3' = \frac{20}{1+x_1} + \frac{10}{1+4x_3}.$$

Taking into account (4.15), we have $x_3 = 1 - x_1$, and

$$T' = \frac{20}{1+x_1} + \frac{10}{5-4x_1}.$$

We trace this function to maximum; let us find its derivative from x_1:

$$\frac{dT'}{dx_1} = \frac{20}{(1+x_1)^2} - \frac{40}{(5-4x_1)^2} = \frac{(5-4x_1)^2\, 20 - (1+x_1)^2\, 40}{(1+x_1)^2\,(5-4x_1)^2}.$$

The derivative dT/dx_1 is converted into zero when it is

converted into zero numerator. Solving the obtained quadratic equation and taking that root which lie/rests between zero and one, we will obtain $x_1 \approx 0.66$.

It is not difficult to directly ascertain that at this point is reached the minimum, but not the maximum of value T'. Since $x_1 = 0.66 < 0.8$, then critical way will be preserved.

Thus, of our example the advantageous redistribution of resources consists of following: from the available supply of the free resources $b_2 = 1$ resource $x_1 = 0.66$, they must be postponed by work a_1, and resources $x_3 = 1 - 0.66 = 0.34$ - to work a_3. In this case, the time of the execution of the complex of works takes the minimum value $T' = 16.29$. The times of the execution of the separate works a_1, a_2 and a_3 will be equal to with respect $t'_1 = 12.05$, $t'_2 = 11.11$, $t'_3 = 4.24$.

Problem 3. There is a complex of works $a_1, a_2, \ldots, a_n$ at times of execution $t_1, t_2, \ldots, t_n$. For this complex is found critical way and it is establish/installed that the minimum time of the execution of complex $T < T_0$, where T_0 - assigned period of execution. It is possible to reduce the rates of the execution of some works with the fact in order the period of the execution of complex to bring to the assigned value T_0; because of this it is possible to obtain the economy of

resources. An increase in the time of the execution of work a_i on τ (i.e. finishing/bringing the time of the execution of work a_i to $t_i+\tau$) frees some resources x_i, which they depend on delay τ:

$$x_i = f_i(\tau).$$

It is required to determine, how one should detain the execution of each work so that the period T_0 would be maintained, and the economy of resources was obtained maximum.

Let us designate τ_i the delay time of work a_i.

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The sum of the times of the execution of the works, which lie on critical way, must not exceed T_0:

$$\sum_{(\text{кр})} (t_i + \tau_i) \leqslant T_0,$$

where the sum, as earlier, it extends only to critical works. It is required to select such nonnegative values of variables τ_i, so that the sum of the freeing resources would reach the maximum:

$$\sum_i f_i(\tau_i) = \max.$$

Stated problem again is related to the class of the problems of nonlinear programming. In the case when speech occurs only about insignificant delays τ_i, sometimes it is possible to reduce it to the problem of linear programming (namely, if function f_i are close to linear in the range of the possible values τ_i, a critical way with

delays it does not vary).

5. Network gliding/planning with the random times of the execution of works. Application/use of ETsVM.

Until recently, examining the problems of planning the complex of works, we were limited to the case when the times of the execution of separate works were to us in accuracy known previously (the so-called determined case). In practice this rarely is as follows: more frequently are encountered the cases when the actual time of the execution of work previously in accuracy is unknown (it is random) and it can strongly differ from its forecast value. The deviation of random variable t_i — the time of the execution of work a_i — from its predetermined value $t_i^{(0)}$ it can be, generally speaking, to both sides - both into large (delay) and in smaller (lead/advance), although in practice the second is encountered much more rarely than the first.

Do arise the following questions:

- Which probability that the actual time of execution of the complex of works T will not exceed assigned magnitude T_0?

- As one should organize the complex of works so that value T

would not exceed given one T_0 with sufficiently high probability?

Let us consider the first question as more idle time (especially because for an answer/response to the second, primarily, must be able to answer the first). Let us assume that the times of the execution of works $t_1, t_2, \ldots, t_n$ represent by themselves random variables with the known laws of distribution. Let us assume for simplicity that these random variables are independent, and their densities equal to

$$f_1(t),\ f_2(t),\ \ldots,\ f_n(t).$$

Is examined the function of these random variables - total time of the execution of entire complex of the works:

$$T=\sum_{(\text{кр})} t_i. \tag{5.1}$$

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Stated problem will be solved, if we can find the function of random number distribution T:

$$F(t)=P(T<t).$$

Then, substituting in it for t value T_0, we find the unknown probability.

Function (5.1) in the general case is fairly complicated, since critical way itself is accidental and depends on those values which take random variables t_i — times of the execution of the separate works: with some values t_i there can be one critical way, with

others - another. However, if we are bounded only to comparatively small deviations of random variables t_i from its nominal values $t_i^{(0)}$ (by so small which critical way remains the same), then problem strongly is simplified. Then in formula (5.1) they figure only several completely specific random variables t_i — times of the execution of critical works. The law of random number distribution T represents by itself in this case nothing else but the composition of the laws of random number distribution t_i, relating to critical works.

Subsequently to us comes to the aid very complexity of plan/layout and the presence on the critical way of many works. We know that with combination of a sufficiently large number of the independent random quantities, distributed according to any laws and congruent in order of dispersions, the law of sum distribution proves to be close to normal (central limit theorem). Therefore, if on critical way stands sufficiently large number of works (let us say, that order 5-6 or it is more), then in practice it is possible to approximately consider value T distributed normally. Its mathematical expectation will be equal

$$m_t = \sum_{(\text{кр})} m_{t_i},$$

where m_{t_i} — mathematical expectation of the time of the execution of the i work, while its root-mean-square deviation:

$$\sigma_t = \sqrt{\sum_{(\text{кр})} \sigma^2_{t_i}},$$

where σ_{t_i} — the root-mean-square deviation of the time of the execution of the i work.

Thus, in this case for the determination of the law of time allocation of the execution of the complex of works there is no necessity to know the laws of the distribution $f_i(t)$ of various times t_i; it suffices to know their mathematical expectations and root-mean-square deviation. If these values are known, the probability of the execution of complex in period T_0 will be located from the known formula

$$P(T<T_0)=\Phi\left(\frac{T_0-m_t}{\sigma_t}\right)+0{,}5, \qquad (5.2)$$

where Φ — a function of Laplace (see appendix, Table 1).

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Example 1. during the execution of the complex of works $a_1, a_2, \ldots, a_n$ critical prove to be the works

$$a_1,\ a_2,\ a_3,\ a_4,\ a_5,\ a_6,$$

the times of execution of which represent by themselves random variables

$$t_1,\ t_2,\ t_3,\ t_4,\ t_5,\ t_6$$

with the mathematical expectations

$$m_{t_1}=10,\ m_{t_2}=20,\ m_{t_3}=10,\ m_{t_4}=5,\ m_{t_5}=7,\ m_{t_6}=10$$

and the root-mean-square deviation:

$$\sigma_{t_1}=1,\ \sigma_{t_2}=1,\ \sigma_{t_3}=0{,}5,\ \sigma_{t_4}=0{,}3,\ \sigma_{t_5}=0{,}5,\ \sigma_{t_6}=1.$$

The probable deviations of the times of the execution of works from their mathematical expectations do not vary critical way. Is assigned the period of the execution of the complex $T_0 = 65$. To find probability that this period will be carried out.

Solution. We have:

$$m_t = 10+20+10+5+7+10=62,$$

$$\sigma_t = \sqrt{1^2+1^2+0,5^2+0,3^2+0,5^2+1^2} = \sqrt{3,59} \approx 1,9.$$

Probability of the execution of the complex of works within period $T_0 = 65$:

$$P(T<65)=\Phi\left(\frac{65-62}{1,9}\right)+0,5=\Phi(1,58)+0,5.$$

Through Table 1 of application/appendix we find: $\Phi(1,58) \approx 0,44$, whence the probability of the execution of complex within period $P(T < 65) \approx 0.94$.

If during random changes in the times t_i can vary critical way itself, the problem of calculating the probability $P(T < T_0)$ hinders. With a comparatively small number of works in complex, this problem can be solved by analytical method, but with their large number calculations become excessively bulky, and in practice it proves to be to conveniently determine these probabilities by the Monte-Carlo method by ETsVM (see Chapter 8). In this case, are developed the values of random times t_i and for each set of the obtained values is determined time T of the execution of the complex of works by that method which is applied for nonaccidental times.

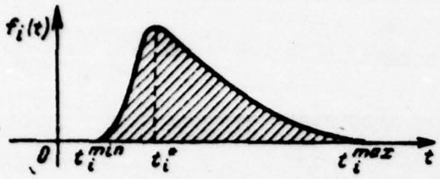

Fig. 10.6.

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After obtaining sufficiently large number N of such realizations, we can directly find mathematical expectation, dispersion and the root-mean-square deviation of random variable T. As concerns the law of distribution, then it in the majority of cases for complex grid/networks proves to be close to normal. Therefore the probability of the fulfilment of plan within period can be calculated according to the same formula (5.1). If there are foundations for considering the law of the distribution of value T not normal (thus, for instance, it is, if scattering the time of execution any one of the critical works sharply exceeds scattering the others), then as approximate value of probability $P(T < T_0)$ it is possible to take frequency of this event in the series of realizations.

It is necessary to note that this type of calculations can be only especially tentative, since in practice usually the laws of distribution $f_i(t)$ are unknown, and their obtaining the statistical data is difficult. At best succeeds in indicating for each time t_i

its most probable value t_i^*, and also rough to consider small ("optimistic") value t_i^{min} and great (" pessimistic") value t_i^{max} (Fig. 10.6). As concerns very distribution $f_i(t)$, it it is necessary to assign sufficiently arbitrarily, on the basis of speculative considerations. For example the fact that curve in Fig. 10.6 has the positive asymmetry (is more stretched to the right, than to the left) reflects that well-known fact that delay in comparison with planned period they can be considerably more than lead/advance.

In conclusion let us pause at one more question, connected with the application/use of ETsVM during network gliding/planning. Usually during the execution of the involved complexs of works, the initially selected plans are not implemented, and it is necessary on operation to reexamine them. In this case, it is extremely convenient to hold all data on complex - both the original plan and incoming information on its disturbance/breakdown - in the memory of ETsVM, which from time to time anew examines job schedule, finds for each moment of time new critical way - "threatening" through life of work —— and optimizes plan/layout, indicating, which precisely works and into any degree one should boost.

The fruitful application/use of a method of network gliding/planning during the organization of complex complexes of works is possible only with the condition of the continuous monitoring of plan/layout and its optimization with the help of ETsVM.

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APPENDIX.

Table 1. Values of the function of Laplace $\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_0^x e^{-\frac{t^2}{2}}dt$

x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$
0,00	0,0000	0,45	0,1736	0,90	0,3159	1,35	0,4115	1,80	0,4641	2,50	0,4938
01	0040	46	1772	91	3186	36	4131	81	4649	52	4941
02	0080	47	1808	92	3212	37	4147	82	4656	54	4945
03	0120	48	1844	93	3338	38	4162	83	4664	56	4948
04	0160	49	1879	94	3264	39	4177	84	4671	58	4951
05	0199			95	3289			85	4678		
06	0239	0,50	0,1915	96	3315	1,40	0,4192	86	4686	2,60	0,4953
07	0279	51	1950	97	3340	41	4207	87	4693	62	4956
08	0319	52	1985	98	3365	42	4222	88	4699	64	4959
09	0359	53	2019	99	3389	43	4236	89	4706	66	4961
		54	2054			44	4251			68	4963
0,10	0,0398	55	2088	1,00	0,3413	45	4265	1,90	0,4713	2,70	0,4965
11	0438	56	2123	01	3438	46	4279	91	4719	72	4967
12	0478	57	2157	02	3461	47	4292	92	4726	74	4969
13	0517	58	2190	03	3485	48	4306	93	4732	76	4971
14	0557	59	2224	04	3508	49	4319	94	4738	78	4973
15	0596			05	3531			95	4744		
16	0636	0,60	0,2257	06	3554	1,50	0,4332	96	4750	2,80	0,4974
17	0675	61	2291	07	3577	51	4345	97	4756	82	4976
18	0714	62	2324	08	3599	52	4357	98	4761	84	4977
19	0753	63	2357	09	3621	53	4370	99	4767	86	4979
		64	2389			54	4382			88	4980
0,20	0,0793	65	2422	1,10	0,3643	55	4394	2,00	0,4772	2,90	0,4981
21	0832	66	2454	11	3665	56	4406	02	4783	92	4982
22	0871	67	2486	12	3686	57	4418	04	4793	94	4984
23	0910	68	2517	13	3708	58	4429	06	4803	96	4985
24	0948	69	2549	14	3729	59	4441	08	4812	98	4986
25	0987			15	3746			2,10	0,4821		
26	1026	0,70	0,2580	16	3770	1,60	0,4452	12	4830	3,00	0,49865
27	1064	71	2611	17	3790	61	4463	14	4838	3,10	49903
28	1103	72	2642	18	3810	62	4474	16	4846	3,20	49931
29	1141	73	2673	19	3830	63	4484	18	4854	3,30	49952
		74	2703			64	4495			3,40	49966
0,30	0,1179	75	2734	1,20	0,3849	65	4505	2,20	0,4861	3,50	49977
31	1217	76	2764	21	3869	66	4515	22	4868	3,60	49984
32	1255	77	2794	22	3888	67	4525	24	4875	3,70	49989
33	1293	78	2823	23	3907	68	4535	26	4881	3,80	49993
34	1331	79	2852	24	3925	69	4545	28	4887	3,90	49995
35	1368			25	3944			2,30	0,4893		
36	1406	0,80	0,2881	26	3962	1,70	0,4554	32	4898	4,00	0,499968
37	1443	81	2910	27	3980	71	4564	34	4904	4,50	499997
38	1480	82	2939	28	3997	72	4573	36	4909	5,00	0,49999997
39	1517	83	2967	29	4015	73	4582	38	4913		
		84	2995			74	4591				
0,40	0,1554	85	3023	1,30	0,4032	75	4599	2,40	0,4918		
41	1591	86	3051	31	4049	76	4608	42	4922		
42	1628	87	3078	32	4066	77	4616	44	4927		
43	1664	88	3106	33	4083	78	4625	46	4931		
44	1700	89	3133	34	4099	79	4633	48	4934		

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Table 2. Values $P_m = \frac{a^m}{m!} e^{-a}$ (Poisson distribution).

m \ a	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0	0,9048	0,8187	0,7408	0,6703	0,6065	0,5488	0,4966	0,4493	0,4066
1	0,0905	0,1638	0,2222	0,2681	0,3033	0,3293	0,3476	0,3595	0,3659
2	0,0045	0,0164	0,0333	0,0536	0,0758	0,0988	0,1217	0,1438	0,1647
3	0,0002	0,0019	0,0033	0,0072	0,0126	0,0198	0,0284	0,0383	0,0494
4		0,0001	0,0002	0,0007	0,0016	0,0030	0,0050	0,0077	0,0111
5				0,0001	0,0002	0,0004	0,0007	0,0012	0,0020
6							0,0001	0,0002	0,0003

m \ a	1	2	3	4	5	6	7	8	9	10
0	0,3679	0,1353	0,0498	0,0183	0,0067	0,0025	0,0009	0,0003	0,0001	0,0000
1	0,3679	0,2707	0,1494	0,0733	0,0337	0,0149	0,0064	0,0027	0,0011	0,0005
2	0,1389	0,2707	0,2240	0,1465	0,0842	0,0446	0,0223	0,0107	0,0050	0,0023
3	0,0613	0,1804	0,2240	0,1954	0,1404	0,0892	0,0521	0,0286	0,0150	0,0076
4	0,0153	0,0902	0,1680	0,1954	0,1755	0,1339	0,0912	0,0572	0,0337	0,0189
5	0,0031	0,0361	0,1008	0,1563	0,1755	0,1606	0,1277	0,0916	0,0607	0,0378
6	0,0005	0,0120	0,0504	0,1042	0,1462	0,1606	0,1490	0,1221	0,0911	0,0631
7	0,0001	0,0037	0,0216	0,0595	0,1044	0,1377	0,1490	0,1396	0,1171	0,0901
8		0,0009	0,0081	0,0298	0,0653	0,1033	0,1304	0,1396	0,1318	0,1126
9		0,0002	0,0027	0,0132	0,0363	0,0688	0,1014	0,1241	0,1318	0,1251
10			0,0008	0,0053	0,0181	0,0413	0,0710	0,0993	0,1186	0,1251
11			0,0002	0,0019	0,0082	0,0225	0,0452	0,0722	0,0970	0,1137
12			0,0001	0,0006	0,0034	0,0126	0,0263	0,0481	0,0728	0,0948
13				0,0002	0,0013	0,0052	0,0142	0,0296	0,0504	0,0729
14				0,0001	0,0005	0,0022	0,0071	0,0169	0,0324	0,0521
15					0,0002	0,0009	0,0033	0,0090	0,0194	0,0347
16						0,0003	0,0014	0,0045	0,0109	0,0217
17						0,0001	0,0006	0,0021	0,0058	0,0128
18							0,0002	0,0009	0,0029	0,0071
19							0,0001	0,0004	0,0014	0,0037
20								0,0002	0,0006	0,0019
21								0,0001	0,0003	0,0009
22									0,0001	0,0004
23										0,0002
24										0,0001

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